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Physical Interpretation of the Electromagnetic Mass

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Abstract. It is well known that the factor 4/3 appears in the theory of a charged particle, that is, there is a discrepancy between the mass determined from the momentum due to the self-field and that defined by the field energy of the particle.

Through extensive examination of the behavior of charged objects, this paper elucidates the physical reason why and how this factor comes out in the theory. It is illustrated with a self-consistent physical picture that there are two types of energy flow: one is transportation of energy itself and the other is translation of stresses. This fact is a direct consequence of the definition of energy, and is the fundamental reason to cause the discrepancy, because the translation of stresses brings forth momentum without mass. Thus, the legitimate treatment of momentum reduces the contradiction automatically.

By resolving major confusions so far occurred in connection with this problem, the significance of the representative endeavors in the past to solve the problem will be appreciated accordingly. The theory presented here is also applicable to other fields than electromagnetic field.

1 Introduction

Around the turn of this century Abraham [1] and Lorentz [2] attempted to make a purely electromagnetic model of a charged particle. However, they obtained the famed result that the momentum of the electron differed by the factor 4/3 from the value which is expected from the rest mass of the electron. Ever since, the endeavors were continued to solve the meaning of this factor or to remove it. With the aid of examples, this paper explores the root of the discrepancy from a wider point of view, manifesting which is the legitimate solution for the problem on the electromagnetic mass.

2 The Point at Issue

For convenience of the later description, the problems on the electromagnetic mass will be briefly summarized along with the attempts so far undertaken to solve them.

2.1 Abraham-Lorentz-Poincaré model

The electron is classically modeled as a particle carrying a spherically symmetric charge distribution. With the assumption that the charge is distributed on the surface of a sphere, the classical radius of the electron $2r_e = e^2/4\pi\varepsilon_0 m_e c^2$ has been determined, so that the electrostatic energy around the electron integrated over the space from radius $r = r_e$ to $r = \infty$ equals to the mass m_e of the electron multiplied by c^2. If the electron moves with a constant velocity of $\boldsymbol{v}$, the momentum of the electron assumes the value of $(4/3)\gamma m_e \boldsymbol{v}$. This discrepancy appears to have been first found by J.J. Thomson [3].

An important suggestion was made by Poincaré [4] to solve this problem. His mathematical theory may be epitomized as follows, using a more compact notation. Employing the four-dimensional space-time representation (ct, x, y, z) with the metric tensor

$$g^{\mu\nu} = \text{diag }(1,\ -1,\ -1,\ -1),$$

the electromagnetic energy-momentum tensor is expressed as

$$T_e^{\mu\nu} = \begin{bmatrix} W & c\boldsymbol{G}^t \\ \boldsymbol{S}/c & -\mathsf{M} \end{bmatrix}, \qquad (\mu, \nu = 0, 1, 2, 3) \tag{1}$$

where

$$W = \frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2/\mu_0,\ \ \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H},\ \ \boldsymbol{G} = \boldsymbol{D} \times \boldsymbol{B},$$

$$\mathsf{M} = \varepsilon_0 \boldsymbol{E}\boldsymbol{E}^t + \boldsymbol{B}\boldsymbol{B}^t/\mu_0 - \mathsf{I}\,\frac{1}{2}(\varepsilon_0 \boldsymbol{E}^2 + \boldsymbol{B}^2/\mu_0)\,.$$

The superscript t denotes that the rows and columns are interchanged, and I the 3-dimensional unit matrix or idem-factor.

Now, in the rest system of co-ordinates of an electron ($\boldsymbol{B} = 0$), the integration of the energy-momentum tensor over the entire space ($r \geq r_e$) yields

$$\int T_0{}^{\mu\nu} \mathrm{d}^3 x_0 = \mathrm{diag}\left(m_e c^2,\ \mathsf{I}\,\frac{m_e c^2}{3}\right), \tag{2}$$

where $m_e c^2 = \int (\varepsilon_0 \boldsymbol{E}_0^2/2)\mathrm{d}^3 x_0$ and the subscript 0 denotes the rest system. The off-diagonal terms have been cancelled because of the spherical symmetry of the electric field of the electron.

Next, we observe the electron moving with velocity of $\boldsymbol{v}$. This is equivalent to the situation that we view it from the co-ordinate system moving with $-\boldsymbol{v}$ relative to the electron. Then the matrix for the Lorentz transformation is written.

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & \gamma \boldsymbol{\beta}^t \\ \gamma \boldsymbol{\beta} & \mathsf{I} + (\gamma - 1)\boldsymbol{\beta}\boldsymbol{\beta}^t/\beta^2 \end{bmatrix}, \tag{3}$$

where

$$\boldsymbol{\beta} = \boldsymbol{v}/c, \quad \gamma = 1/\sqrt{1-\beta^2}.$$

With this matrix the energy-momentum tensor in Eq.(2) is transformed to

$$T_e^{\mu\nu} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} T_0{}^{\mu'\nu'}, \tag{4}$$

from which we find the energy and momentum of the electron as

$$\int T_e{}^{0\nu} \mathrm{d}^3 x = \gamma\,\left(m_e c^2 + \frac{1}{3} m_e v^2,\ \frac{4}{3} m_e \boldsymbol{v}^t c\right),$$

where use was made of the relation $\mathrm{d}^3 x_0 = \gamma \mathrm{d}^3 x$. There appears thus the so-called anomalous factor 4/3 in momentum.

Toward the solution of this problem, Poincaré introduced the cohesive forces to prevent the electron from exploding because of the repulsive Coulomb forces among charges. The energy tensor for the cohesive forces is given by

$$\int T_p{}^{\mu\nu} \mathrm{d}^3 x_0 = \mathrm{diag}\left(m_p c^2,\ -\mathsf{I}\,\frac{m_e c^2}{3}\right),$$

where m_p is the non-electromagnetic mass of "matter" to provide the cohesive forces. Adding this tensor to that of Eq.(2) and transforming it through the Lorentz matrix (3) yields

$$\int \Lambda^0{}_{\mu'}\Lambda^\nu{}_{\nu'}\,(\,T_0{}^{\mu'\nu'} + T_p{}^{\mu'\nu'}\,)\,\mathrm{d}^3x \;=\gamma\,[\,(m_e+m_p)c^2,\;\;(m_e+m_p)\boldsymbol{v}^t c]. \tag{5}$$

This shows that the electron has acquired the covariance of energy and momentum under the Lorentz transformation.

2.2 Einstein's model

While Poincaré made no mention of the origin of the cohesive forces, Einstein considered the possibility that such cohesive forces are generated from gravity [5].

The gravitational equation is expressed as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa\, T^{(m)}{}_{\mu\nu},$$

where $R_{\mu\nu}$ denotes the contracted Riemann tensor of curvature, R the scalar of curvature formed by repeated contraction, κ Einstein's constant of gravitation, and $T^{(m)}_{\mu\nu}$ the energy tensor of matter. In order to apply this equation to the construction of the electron, he rebuilt (modified) the above equation by substituting $T^{(m)}{}_{\mu\nu}$ with the electromagnetic energy tensor $T^0{}_{\mu\nu}$. Maxwell's equations were postulated to remain valid. Forming the divergence of the gravitational equation, he obtained

$$F_{\mu\nu}j^\nu - \frac{1}{4\kappa}\frac{\partial R}{\partial x^\mu} = 0,$$

where j^ν is the four-current density. The relation above implies that the Coulomb repulsive forces are held in equilibrium with gravitational (negative) pressure. The $R/4\kappa$ is regarded as the potential energy of gravitational forces.

The possibility of a theoretical construction of matter out of gravitational field and electromagnetic field alone seems very promising. For the spherically symmetric fields, however, the number of the equations is fewer than that of the unknowns, so that it is not capable of providing a definitive answer, Einstein added.

2.3 Fermi-Kwal-Rohrlich treatment

In line with Fermi [6] and Kwal [7], but independent of them Rohrlich claimed to define the electromagnetic energy-momentum vector as [8], [9]

$$R^\mu = \int T_e^{\mu\nu} n_\nu \mathrm{d}\sigma. \quad (\mu,\nu = 0,1,2,3) \tag{6}$$

He explained that $n_\nu = \gamma\,[\,1, -\boldsymbol{\beta}^t\,]$ is the time-like unit vector normal to the surface directed into the future light cone, and that $\mathrm{d}\sigma(= \mathrm{d}^3 x_0)$ is an invariant infinitesimal element of a plane hyper-surface.

Now, if we insert Eq.(1) or (4) into (6), we get the covariant expression

$$R^\mu = \gamma\ [m_e c^2,\ m_e \boldsymbol{v}^t c],$$

as he claimed.

Against this definition of the covariant four-vector, Boyer argued [10] that the non-electromagnetic forces, the Poincaré stresses, are essential to the stability of the electron, and that the factor 4/3 is not erroneous. In his discussion he used the model that a thin spherical shell of mechanical mass and charge is sent inwards from spatial infinity to form the classical electron with cohesive forces. He added his belief to the effect that the re-definition of the electromagnetic momentum apart from the Poynting vector is an error.

Concerning this Rohrlich-Boyer controversy [11], Campos and Jiménez tried to clarify the physical content of both points of view [12], citing Schwinger's derivation of two energy-momentum four-vectors.

2.4 Schwinger's classical electron models

In the paper dedicated to Dirac, Schwinger wrote [13] that there are two values for electromagnetic mass. One is the electrostatic energy divided by c^2 : m_e, and the other is electromagnetic momentum divided by velocity:

$$m_e^{(s)} = (4/3) m_e.$$

By adjusting to these two masses, he exhibited two interesting models of spherically symmetric charge distributions that possess conserved (divergence-less) energy tensors. That is, he introduced two kinds of mechanical stress tensors to secure stable charge distributions, leading to the energy-momentum covariance as four-vectors. These may be regarded as two examples of embodiment of the Poincaré model.

There are numerous papers and books that add comments on this issue [14]. Moylan's paper is a recent one, deducing the factor 4/3 in an elementary way [15].

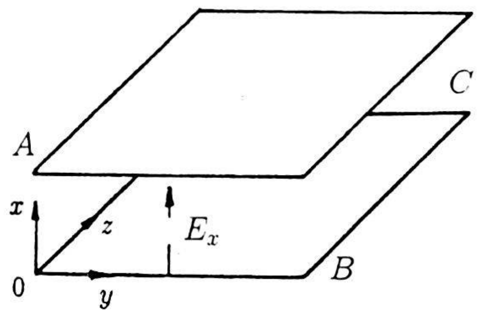

Figure 1: A coupled system of field and matter forming a condenser.

3 Nature of the Electromagnetic field

As it is not admissible that we have two values of electromagnetic mass, the kernel of the problem is to seek for the fundamental reason why the two definitions of mass give us different values.

To unravel this riddle, we will first investigate the characteristic of electrostatic field of simple form. Fig.1 shows a pair of plain conductors in parallel constituting a charged condenser, together with the co-ordinate system in it. If the area of the electrodes is sufficiently wide compared with the spacing A, the electric field E_x is uniform. Within the condenser there is electromagnetic energy and hence electromagnetic mass

$$m = \varepsilon_0 E_x^2/2c^2$$

per unit volume, in accordance with the commonly accepted law of physics.

3.1 Momentum in parallel with field

Let us suppose that the condenser is driven with a uniform velocity $\boldsymbol{v} = \hat{x}v$ in the direction parallel to the electric field, where $\hat{x}$ is the unit vector in the x direction. As the electrostatic field must move along with the condenser, it may be expected that there should be electromagnetic momentum density upwards in the positive x direction being equal to $\gamma^2 m\boldsymbol{v}$. [1]

However, according to the Lorentz transformation, the electric field E_x within the condenser does not change, nor magnetic field is generated ($E'_x = E_x$, $H'_x = 0$), as long as

[1]The γ^2 denotes the increase of mass and the Fitzgerald-Lorentz contraction of length (increase of density) in the x direction.

the field is uniform (the area of the condenser is wide enough). It follows then that there is no Poynting's vector and hence no flow of electromagnetic energy. That is, there is no electromagnetic momentum along the x axis in parallel with the electric field E_x. If the electromagnetic mass is to be defined by way of momentum, it must be zero in this case. This seems to be a paradox. Or, is the Poynting vector erroneous, as various researchers suggested ? [2] [17].

To interpret the meaning of this situation, we should take a wider view. In order for the electrostatic field to maintain its existence, there must be a repulsive force to support the electrodes of the condenser against the attractive force of the field. For, if the supporter or insulating separator of the electrodes (not shown specifically in Fig.1) is removed, then the space between the electrodes is diminished to naught. This is an example of the Earnshaw theorem.

Elementary electromagnetics tells that the attractive force between the electrodes is equal to $\varepsilon_0 E_x^2/2$ per unit area. This force does work on the lower electrode giving it an amount of energy $\gamma^2(\varepsilon_0 E_x^2/2)\, v$ per unit area per unit time, when the condenser moves upwards in the positive x direction. On the other hand, the upper electrode does work giving the same amount of energy to the field. In compensation for the energy loss of the upper electrode, there must be flow of energy supplied from the lower to the upper electrode $\gamma^2(\varepsilon_0 E_x^2/2)BC\ \hat{x}v$ through the separator, where BC is the area of the electrodes. Since the amount of this energy flow corresponds to the momentum of $\gamma^2(\varepsilon_0 E_x^2/2c^2)BC\ \hat{x}v$ upward, the condenser as a whole satisfies the covariance of energy and momentum. The mass of the condenser has been excluded for simplicity.

3.2 Momentum in perpendicular to field

Next consider the case where the condenser is driven in the direction of positive z axis perpendicular to the field. Through the Lorentz transformation, the electromagnetic field within the condenser is transformed to ($\boldsymbol{v} = \hat{z}v$)

$$E'_x = \gamma E_x, \quad H'_y = \gamma v \varepsilon_0 E_x,$$

[2]A little after Poynting's proposal (1884), Hertz criticized [16] that any quantity whose divergence vanishes can be added to $\boldsymbol{E} \times \boldsymbol{H}$ without violating Poynting's theorem. It can be shown, however, that the Poynting vector correctly represents the electromagnetic energy flow density, as will be published elsewhere by the present author. Another verification in physical terms will be shown in the later section of this paper.

so that the Poynting vector

$$\boldsymbol{E} \times \boldsymbol{H} = \hat{z} E'_x H'_y = \hat{z} \gamma^2 v \varepsilon_0 E_x^2 \tag{7}$$

comes into being in the positive z direction. From this it follows that there is electromagnetic momentum density

$$\hat{z} E'_x H'_y / c^2 = 2\gamma^2 m \boldsymbol{v}. \tag{8}$$

What does the factor 2 of this momentum mean ?

In accordance with electromagnetics, there is electrostatic pressure $\varepsilon_0 E_x^2/2$ in the lateral direction. In other words, the field has a tendency to expand perpendicularly to it, so that the side aperture and hence the electrodes receives force from the field [3]. Since this force is equal to $(\varepsilon_0 E_x^2/2)\ AB$, the front end $(z = C)$ of the electrodes receives energy $\gamma^2(\varepsilon_0 E_x^2/2) v\, AB$ per unit time, while the rear end $(z = 0)$ of the electrodes gives the same amount of energy to the field.

This implies that there must be backward energy flow through the electrodes from the front to the rear. This reduces the forward momentum (8) of the field by one half. In this manner, the energy-momentum covariance is recovered in combination of the field and the electrodes with the separator.

3.3 Field confined in a cavity

As a more methodical example [4], consider the case where electromagnetic field is confined perfectly in a cavity made up of metallic walls at $y = 0,\ B$ and $z = 0,\ C$ in Fig.1. When the fundamental mode with electric field

$$E_x = E \sin k_y y\ \sin k_z z\ \cos \omega t,$$

is excited in the cavity, it is also demonstrated that the cavity together with this field exhibits the covariance of energy and momentum. However, the average momentum of the field in the z direction has the value

$$\gamma M \boldsymbol{v}(1 + k_z^2/k^2), \qquad (\boldsymbol{v} = \hat{z} v) \tag{9}$$

[3] Near the edges of the electrodes the field is deformed, so that the longitudinal or z component of the field draws the electrodes toward outside. If magnetic walls are placed hypothetically at the ends of the electrodes so that no fringing effect occurs, then the walls accept the pressure $\varepsilon_0 E_x^2/2$ uniformly.

[4] An excerpt from the Japanese version.

which is compensated for by the momentum through the cavity walls $-\gamma M\boldsymbol{v}(k_z^2/k^2)$. The M stands for the total energy of the field divided by c^2, and k and k_z the wave numbers in free space and in the z direction, respectively.

4 Electrodynamics of Charged Objects

4.1 General consideration

In the previous section, a simple model has been physically examined which consists of electromagnetic field and matter. We are now in a position to treat such systems rigorously from a more general and unified point of view.

In order for a coupled system of electromagnetic field with matter to remain in equilibrium, the stresses must vanish as a whole. That is, the stability condition [5] [14], [18].

$$\int (M_0^{mn} + T_p^{mn})\, \mathrm{d}^3x_0 = 0, \qquad (m, n = 1, 2, 3) \tag{10}$$

should hold, where M_0^{mn} and T_p^{mn} are the stress tensors of the electromagnetic field and of the matter respectively in the co-ordinate system at rest [6]. It should be remarked here that the stress tensor T_p^{mn} includes not only the Poincaré cohesive forces but also repulsive forces as in the case of the condenser. The integration is performed over the space including the coupled system.

Now, the energy tensor pertaining to the electrostatic field in the rest frame is expressed as

$$T_0^{\mu\nu} = \begin{bmatrix} W_0 & \mathbf{o} \\ \mathbf{o} & -\mathsf{M}_0 \end{bmatrix}, \qquad (\mu, \nu = 0, 1, 2, 3) \tag{11}$$

where $W_0 = \varepsilon_0 \boldsymbol{E}_0^2/2$ and $\mathbf{o}$ stands for a three-dimensional zero vector. The Maxwell stress tensor is expressed as $\mathsf{M}_0 = [M_0^{mn}] = \varepsilon_0 \boldsymbol{E}_0 \boldsymbol{E}_0 - \varepsilon_0 \boldsymbol{E}_0^2/2$. If this is viewed from a frame

[5] This is derivable through the four-dimensional integration of the conservation law of energy and momentum, $\partial T^{\mu\nu'}/\partial x^{\nu'} = 0$, where $T^{\mu\nu}$ is the energy-momentum tensor of the system .

[6] In this paper, the energy-momentum tensors are generally designated by $T^{\mu\nu}\,(\mu, \nu = 0, 1, 2, 3)$, the time part T^{00} being the energy and the negative of the space part $-T^{mn} = T_s^{mn}\,(m, n = 1, 2, 3)$ being the stresses, with a subscript s to indicate their kind. Especially for electromagnetic field, W is used for T_e^{00} and M for the dyadic $[-T^{mn}]$, discriminating between the rest and the moving systems with or without subscript 0.

moving with $-\boldsymbol{v}$, it takes the form of Eq.(4):

$$T^{\mu\nu} = \Lambda\,[\,T_0^{\mu\nu}]\,\Lambda^t = \gamma^2 \begin{bmatrix} W_0 - \boldsymbol{\beta}\cdot\mathsf{M}_0\cdot\boldsymbol{\beta} & \boldsymbol{\beta}W_0 - \boldsymbol{\beta}\cdot\mathsf{M}_0\cdot\mathsf{L} \\ W_0\boldsymbol{\beta} - \mathsf{L}\cdot\mathsf{M}_0\cdot\boldsymbol{\beta} & \boldsymbol{\beta\beta}W_0 - \mathsf{L}\cdot\mathsf{M}_0\cdot\mathsf{L} \end{bmatrix}, \tag{12}$$

where

$$\gamma\mathsf{L} = \mathsf{I} + (\gamma-1)\boldsymbol{\beta\beta}/\beta^2.$$

For simplicity of expression, the dyadic notation with three-vectors is employed hereafter, omitting the superscript t (transposed) within the matrix. The first row or column shows that the electromagnetic momentum does not transform covariantly with the energy.

On the other hand, the energy tensor of the combined system integrated over the volume transforms to

$$\int \Lambda\,[\,T_0^{\mu\nu} + T_p^{\mu\nu}\,]\,\Lambda^t\,\mathrm{d}^3x = \gamma \begin{bmatrix} 1 & \boldsymbol{\beta} \\ \boldsymbol{\beta} & \boldsymbol{\beta\beta} \end{bmatrix} \int (W_0 + m_p c^2)\,\mathrm{d}^3x_0, \tag{13}$$

where the stability condition (10) has been employed. The $m_p = T_p^{00}/c^2$ is the density of matter accompanied with the generalized Poincaré forces. The time component 1 and the space-time component $\boldsymbol{\beta}$ in the tensor above indicate the energy-momentum covariance of the coupled system.

This general formulation is applied first to the condenser to confirm the foregoing physical consideration in terms of mathematics.

4.2 Charged condenser

(Moving in parallel with the field) In case of the condenser moving along the x axis in the previous section, the spatial integration of the energy tensor of the field (12) is calculated as

$$\int \Lambda\,[T_0^{\mu\nu}]\,\Lambda^t\,\mathrm{d}^3x = \int \mathrm{d}^3x_0 W_0 \begin{bmatrix} 1 & \mathbf{o} \\ \mathbf{o} & -\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \end{bmatrix}, \tag{14}$$

with the aid of the auxiliary relations

$$\boldsymbol{\beta} = \hat{x}v/c, \qquad W_0 = \varepsilon_0 E_x^2/2, \qquad \mathsf{M}_0 = W_0\,(\hat{x}\hat{x} - \hat{y}\hat{y} - \hat{z}\hat{z}).$$

The result (14) is unchanged (in form) from the integral of the energy tensor (11) in the rest frame. This is a reflection of the fact that, in case of the parallel motion with the field, the field does not change under the Lorentz transformation. The space-time component shows that there is no electromagnetic momentum, and the space part that there are tension in parallel with and pressure in perpendicular to E_x, assuming the same values as in the rest frame.

The total energy tensor of the charged condenser is equal to Eq.(13) with substitution of $\boldsymbol{\beta} = \hat{x}v/c$ and $\int m_p \mathrm{d}^3x_0$ = (mass of the condenser). The space-time component shows that the system has the momentum

$$\gamma \boldsymbol{v} \int \left(\frac{W_0}{c^2} + m_p \right) \mathrm{d}^3x_0,$$

as should be expected. Even though the field does not convey the electromagnetic momentum as seen from Eq.(14), the equation above tells that the required amount of energy $W_0 = \varepsilon_0 E_x^2/2$ (integrated over the space) is conveyed mechanically through the separator of the electrodes of the condenser.

(Moving in perpendicular to field) The energy tensor (12) for the field $\hat{x}E_x$ moving with velocity $\boldsymbol{v} = \hat{z}v$ assumes the form

$$\int \Lambda \left[T_0^{\mu\nu} \right] \Lambda^t \, \mathrm{d}^3x = \int \mathrm{d}^3x W_0 \cdot \begin{bmatrix} \gamma^2(1+\beta^2) & 2\gamma^2\boldsymbol{\beta} \\ 2\gamma^2\boldsymbol{\beta} & -\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}\gamma^2(1+\beta^2) \end{bmatrix}, \tag{15}$$

where $\boldsymbol{\beta} = \hat{z}v/c$. In this case, the momentum of the field is just twice the value of $\gamma\boldsymbol{v}\int W_0 \, \mathrm{d}^3x_0/c^2$, different from what is expected from kinematics of matter ($\mathrm{d}^3x = \mathrm{d}^3x_0/\gamma$). As is obvious from Eq.(13), the charged condenser as a whole meets the covariance of energy and momentum, indicating that there is backward momentum $-\gamma\boldsymbol{v}\int W_0 \, \mathrm{d}^3x_0/c^2$ to compensate for this doubled momentum, as explained in physical terms previously.

4.3 Assortment of momentum

To interpret the behavior of momentum of a charged object in more detail, we write down the space-time components of the energy-momentum tensor [the integrand of Eq.(12) or (13)] as

$$(\text{momentum density}) = \gamma^2(W_0\boldsymbol{\beta} - \mathsf{L}\cdot\mathsf{M}_0\cdot\boldsymbol{\beta} + m_pc^2\boldsymbol{\beta} - \mathsf{L}\cdot\mathsf{T}_p\cdot\boldsymbol{\beta})/c, \tag{16}$$

disregarding the stability condition (10) tentatively. This equation tells that there are four sorts of momentum. The first term is the direct transportation of electromagnetic mass W_0/c^2, while the second is the momentum caused by the Maxwell's stress tensor M_0. The third term is the transportation of mass of the matter, while the last one comes from the stress tensor T_p of the matter.

The meaning of the description above will become clearer, if we examine the electrodynamics of the condenser in the concrete. Noting that $W_0 = \varepsilon_0 E_x^2/2$, the first term of Eq.(16) is equal to $\gamma^2(\varepsilon_0 E_x^2/2c^2)\boldsymbol{v}$, which means that the electromagnetic mass per unit volume $\varepsilon_0 E_x^2/2c^2$ moves with the velocity $\boldsymbol{v}$ along with the condenser, *in either case* of $\boldsymbol{v} = \hat{x}v$ or $\boldsymbol{v} = \hat{z}v$. This fact is very simple but essentially important.

(Parallel motion with the field) In the case of motion in parallel with the field ($\boldsymbol{v} = \hat{x}v$), the second term

$$-\gamma^2 \mathsf{L} \cdot \mathsf{M}_0 \cdot \boldsymbol{\beta}/c = -\gamma^2(\varepsilon_0 E_x^2/2c^2)\boldsymbol{v}$$

is just equal to the negative of the first ($\varepsilon_0 E_x^2/2 = W_0$). This is the very reason why the field had no momentum at all, when it moved in the direction parallel to it. This indicates that *stress produces momentum when it moves.*

The physical reason why stress in motion has momentum is explained as follows: Consider a hypothetical plane perpendicular to the field E_x. According to electromagnetics, the upper surface of this hypothetical plane is pulled upwards by the tension (Maxwell's stress) $\varepsilon_0 E_x^2/2$ per unit area, while the lower surface is pulled downwards by the same magnitude [7]. When this plane moves upwards in the positive x direction, work is done on the upper surface, so that the upper field gives it energy $\gamma^2(\varepsilon_0 E_x^2/2)v$ per unit area per unit time. On the other hand, the lower surface of the hypothetical plane does work against the downward force due to the field, so that the plane delivers the energy $\gamma^2(\varepsilon_0 E_x^2/2)v$ to the lower field per unit time. If these phenomena are looked from outside, there is an energy flow downwards through the plane in the negative x direction due to the $\hat{x}\hat{x}$ component of the Maxwell stresses. Thus, the tensional stress behaves as if it had negative energy, when it is set in motion.

(Perpendicular motion to the field) In the case of motion perpendicular to the field ($\boldsymbol{v} = \hat{z}v$),

[7] The explanation below may be more realistic. If an infinitesimally thin plane of conductor is inserted perpendicularly to the field, the field is not affected on account of it. However, there will appear positive charges on the upper surface of the conductor, while negative charges on the lower surface of it. These charges are pulled upwards and downwards by the upper and lower fields, respectively. If we notice that the plane conductor can be replaced by any other substance without affecting the charge distribution, it may now be understood intuitively that there is tensional force across the plane, the upper and lower charges being stuck together to be cancelled out.

the second term of the space-time component in Eq.(16)

$$-\gamma^2 \mathsf{L}\cdot\mathsf{M}_0\cdot\boldsymbol{\beta}/c = \gamma^2(\varepsilon_0 E_x^2/2c^2)\,\boldsymbol{v}$$

is equal to the first term in magnitude, so that the resultant momentum becomes twice what might be expected from ordinary kinematics. As there are pressure equal to $W_0 = \varepsilon_0 E_x^2/2$ perpendicular to the field, the forward momentum has been summed up to be $\gamma 2\boldsymbol{\beta}\int W_0\ \mathrm{d}^3x_0/c$ in Eq.(15). The physical reason why the pressure brings forth forward momentum is explained in an analogous way as above, interchanging the signs of the stresses.

4.4 Two types of (electromagnetic) momentum — a physical picture of the Poynting vector —

Electrostatic field $\boldsymbol{E}_0$ uniformly moving with velocity $\boldsymbol{v}$ is transformed to

$$\boldsymbol{E} = \gamma\boldsymbol{E}_0 - (\gamma-1)\boldsymbol{E}_0\cdot\boldsymbol{v}\boldsymbol{v}/v^2, \qquad \boldsymbol{H} = -\gamma\varepsilon_0\boldsymbol{E}_0\times\boldsymbol{v},$$

according to the Lorentz transformation. After some manipulation, the Poynting vector is rewritten in the form

$$\boldsymbol{E}\times\boldsymbol{H} = \gamma^2 W_0\boldsymbol{v} - \gamma^2\mathsf{L}\cdot\mathsf{M}_0\cdot\boldsymbol{v}. \tag{17}$$

This is identical with the space-time component of Eq.(12) multiplied by c, as it should be.

The expression (17) tells us that there are two types of electromagnetic energy flow. The one is the transportation of energy itself, and the other is the translational motion of stresses. The physical reason why the stress in motion produces energy flow is as illustrated just before in the simple cases where the stress tensor is diagonal. The following consideration may be of help to a deeper understanding of this phenomenon.

In the field of acoustics, the energy (power) flow density is defined to be $-\boldsymbol{v}\cdot\mathsf{T}$[19], [20] [8], where $\boldsymbol{v}$ and T are the velocity and the stress tensor of the medium, respectively. But, in this definition in acoustics, the energy flow $\rho c^2\boldsymbol{v}$ due to the transportation of the medium itself is not taken into account, where ρ denotes the mass density of the medium.

[8]The proof is given in the references. The physical meaning is as follows. On the plane normal to $\hat{n}$ in the medium, there is force $\mathsf{T}\cdot\hat{n}$ per unit area exerted toward $\hat{n}$ according to the definition of the stress tensor T. If the plane moves with velocity $\boldsymbol{v}$, the surface $\hat{n}$ receives energy $\boldsymbol{v}\cdot\mathsf{T}\cdot\hat{n}$ per unit time, while the opposite side gives the same amount of energy to the medium in the $-\hat{n}$ direction. Therefore, energy is delivered through the plane from $+\hat{n}$ to $-\hat{n}$ side per unit area per unit time. Since $\hat{n}$ is arbitrarily chosen, $-\boldsymbol{v}\cdot\mathsf{T}$ represents energy flow in the direction of this vector. In case that the T is symmetric as usual, $\boldsymbol{v}\cdot\mathsf{T}\cdot\hat{n} = \hat{n}\cdot\mathsf{T}\cdot\boldsymbol{v}$. To be rigorous (relativistically), $-\boldsymbol{v}\cdot\mathsf{T}$ should be replaced by $-\gamma^2\boldsymbol{v}\cdot\mathsf{T}\cdot\mathsf{L}$.

Conversely, if this concept is applied to the case of electrostatic field, the total flow of energy is inferred to be given by $W_0\boldsymbol{v} - \mathsf{M}_0 \cdot \boldsymbol{v}$, since the energy density ρc^2 corresponds to $W_0 = \varepsilon_0 \boldsymbol{E}_0^2/2$. This is no more than the non-relativistic limit of Eq.(17), where $\gamma \to 1$ and $\mathsf{L} = \gamma^{-1}\{\mathsf{I} + (\gamma - 1)\hat{v}\hat{v}.\} \to \mathsf{I}$.

Unless the stress tensor is of special type, the direction of energy flow deviates from that of the velocity $\boldsymbol{v}$ in general. If M_0 has only a $\hat{v}\hat{v}$ component $M_{vv}\hat{v}\hat{v}$ with $\hat{v} = \boldsymbol{v}/v$, then the second term of Eq.(17) becomes

$$-\gamma^2 \mathsf{L} \cdot \mathsf{M}_0 \cdot \boldsymbol{v} = -\gamma^2 M_{vv} \boldsymbol{v}. \tag{18}$$

That is, the tension (positive stress) produces negative momentum in the direction of $\boldsymbol{v}$, whereas the pressure (negative stress) produces positive momentum. Stated intuitively, the tension brings forth momentum as if it were negative mass, while pressure does as if positive mass, although the stress itself has no mass [9].

4.5 Circulation of energy

An isolated physical system without interaction from outside satisfies the stability condition (10). When this system is set in motion with $\boldsymbol{v}$, it will have momentum $\gamma M \boldsymbol{v}$ consistent with the energy-momentum covariance, where M is the mass of the system. Provided that there is no stress within the system, each part of the system satisfies the covariance too, because there is no interaction among parts. However, if there is stress inside the system, the stress generates momentum, as explained just before. The magnitude and the orientation of the momentum are determined by the dot product (contraction) of the stress dyadic (tensor) with the velocity. However, as long as the stability condition (10) is satisfied, the integration of the momentum due to stresses over the system vanishes, because

$$\frac{1}{c^2}\int \gamma^2 \mathsf{L} \cdot (\mathsf{M}_0 + \mathsf{T}_p)\, \mathrm{d}^3x \;\cdot \boldsymbol{v} = 0. \tag{19}$$

This equation, on the other hand, implies that there is a circulation of momentum within the system, that is, if there is momentum equal to $\int \gamma^2 \mathsf{L} \cdot \mathsf{M}_0 \mathrm{d}^3x \cdot \boldsymbol{v}/c^2$ somewhere in the system, then there must be backward momentum $-\int \gamma^2 \mathsf{L} \cdot \mathsf{T}_p \mathrm{d}^3x \cdot \boldsymbol{v}/c^2$ to cancel the former.

In the example of the condenser moving in parallel with the field E_x, there was momentum downwards ($-x$ direction) due to the tension in the field, which was just cancelled by the

[9] The energy which has been required to cause the stress is included in the time component of the energy tensor, so that stress itself has no mass.

momentum upwards generated by the pressure in the separator of the electrodes of the condenser. As a resultant, only the momentum conveying the field and the mass of the condenser contributed to the net momentum of the system. This is the physical reason why the combined system in equilibrium exhibits the covariance of energy and momentum, whereas the field or the matter does not do separately because of the stresses which are not cancelled locally.

In case of the condenser moving perpendicular to the field, a similar argument is applicable making exchange between tension and pressure.

4.6 Spherically charged bodies

For a spherically charged body with q, the energy flow caused by stresses, the second term of Eq.(17), is written as

$$-\gamma^2 \mathsf{L}\cdot\mathsf{M}_0\cdot\boldsymbol{v} \Rightarrow -(\varepsilon_0\boldsymbol{E}_0\boldsymbol{E}_0-\varepsilon_0\boldsymbol{E}_0^2/2)\cdot\boldsymbol{v} = -\{\varepsilon_0 E^2(r)/2\}\,(2\hat{\theta}\sin\theta+\hat{z})\,v \tag{20}$$

in the non-relativistic limit for brevity, where

$$\boldsymbol{E}_0 = \hat{r}E(r) = \hat{r}q/4\pi\varepsilon_0 r^2 \ \text{ and } \ \boldsymbol{v} = \hat{z}v. \tag{21}$$

The $\hat{\theta}$ is the unit base vector in the direction of zenith angle of the spherical co-ordinate system. If we add the direct transportation of energy $\boldsymbol{v}\varepsilon_0 E^2(r)/2$ [the first term of Eq.(17)] to Eq.(20), we get the net energy flow

$$-\hat{\theta}\ \varepsilon_0 E^2(r) v\sin\theta \quad (= \boldsymbol{E}\times\boldsymbol{H}).$$

This is equal to the Poynting vector for a spherically charged body in motion in the non-relativistic limit [10]. The expression above indicates that the electromagnetic energy flows along the surface of the spherical body in the negative direction of the zenith angle θ, which is sketched in Fig.2. The energy spouts out from the rear surface and sinks down onto the front surface of the body. Within the body, energy flows backwards in the negative direction of $\hat{z}$. This flow of energy is caused by the tension inside the body, viz. cohesive forces to prevent the repulsion of the charges.

[10] When the Coulomb field (21) moves with velocity $\boldsymbol{v}$, the magnetic field is generated as $\boldsymbol{H} = \boldsymbol{v}\times\varepsilon_0\boldsymbol{E}$ in an elementary way, so that $\boldsymbol{E}\times\boldsymbol{H} = \varepsilon_0\boldsymbol{E}_0\times(\boldsymbol{v}\times\boldsymbol{E}_0)$ giving the same result.

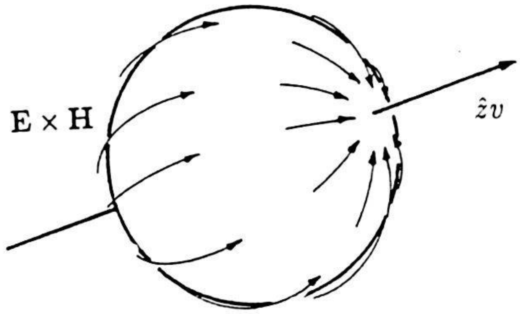

Figure 2: Energy flow around a spherically charged body in motion.

Now, returning to Eq.(20), the electromagnetic momentum caused only by Maxwell's stress is integrated over the entire space exterior to the body to yield

$$-\frac{v}{c^2}\int_{r_q}^{\infty}\frac{1}{2}\varepsilon_0 E^2(r) r^2 \mathrm{d}r \int_0^{\pi}(2\hat{\theta}\sin\theta+\hat{z})\sin\theta \mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi = \frac{1}{3}\boldsymbol{v}\frac{1}{c^2}\int_{r_q}^{\infty}\frac{1}{2}\varepsilon_0 E^2(r)\mathrm{d}^3x_0 \qquad (\mathrm{d}^3x_0 = 4\pi r^2\mathrm{d}r).$$

where use was made of the base vector relation $\hat{\theta} = -\hat{z}\sin\theta + \hat{\rho}\cos\theta$.

The factor 1/3 above is intuitively deducible as below. The resultant electromagnetic momentum due to stresses is given by the vectorial sum over the entire field. The spherically symmetric field is divided equally into three components along x, y and z axes, each having momentum contribution of 1/3 ($\varepsilon_0\overline{E_x^2}/2$:pressure), 1/3 ($\varepsilon_0\overline{E_y^2}/2$:pressure) and −1/3 ($\varepsilon_0\overline{E_z^2}/2$:tension) along the $\hat{v} = \hat{z}$ direction [11]. Thus, the factor $1/3 + 1/3 - 1/3 = 1/3$ arises as the extraneous momentum for spherically symmetric charge distribution, which is added to the direct energy transportation with the factor of unity. Hence, the factor $1/3 + 1 = 4/3$ is characteristic of the electromagnetic field with spherical symmetry.

5 Relativistic Invariance as a Four-vector

As was illustrated earlier, the electrostatic field cannot exist without forces of non-electromagnetic origin. Electrostatic field needs its supporter to survive. The supporter may be a charge itself and/or charged electrodes (with separator). In consequence, the electrostatic field behaves as a stable physical entity only in conjunction with non-electromagnetic forces, the energy and momentum of the combined system thereby constituting a four-vector. Either one of the electromagnetic field or the matter does not satisfy the covariance of energy and momentum separately.

[11] $\overline{E_x^2} = \overline{E_y^2} = \overline{E_z^2}$, $\overline{E_x^2} + \overline{E_y^2} + \overline{E_z^2} = \overline{E^2}$, where the over-line denotes average.

5.1 Meaning of Rohrlich's vector

As was described in the introductory section, Rohrlich presented a four-vector (6) as a covariant definition of electromagnetic energy and momentum. Though his explanation is somehow abstract, the meaning of his vector becomes more lucid, if we take a little different point of view. The electromagnetic energy tensor $T^{\mu\nu}$ in his expression is the Lorentz transformation of the same in the rest frame (11):

$$T_e^{\mu\nu} = \Lambda \begin{bmatrix} W_0 & \mathsf{o} \\ \mathsf{o} & -\mathsf{M}_0 \end{bmatrix} \Lambda^t,$$

and, the vector n_ν is expressed as the inverse Lorentz transformation of the time-like unit vector

$$n_\nu = \Lambda^{-1} \begin{bmatrix} 1 \\ \mathsf{o} \end{bmatrix}.$$

Insertion of the above two equations into (6) yields

$$R^\mu = \int \Lambda \begin{bmatrix} W_0 & \mathsf{o} \\ \mathsf{o} & -\mathsf{M}_0 \end{bmatrix} \begin{bmatrix} 1 \\ \mathsf{o} \end{bmatrix} \mathrm{d}^3x_0 = \int \Lambda \begin{bmatrix} W_0 \\ \mathsf{o} \end{bmatrix} \mathrm{d}^3x_0 = \gamma \begin{bmatrix} 1 \\ \boldsymbol{\beta} \end{bmatrix} \int W_0 \mathrm{d}^3x_0, \tag{22}$$

where use was made of the property that the matrix for the Lorentz transformation is symmetric ($\Lambda^t = \Lambda$). In words, his vector is no more than the Lorentz transformation of the time component $\int W_0 \mathrm{d}^3x_0$ which was extracted from the energy-momentum tensor $T_0^{\mu\nu}$ by way of the time-like vector $[\,1,\ \mathsf{o}\,]$. Formally the space part or stress components M_0 are included in $T^{\mu\nu}$ of Eq.(6), but substantially the M_0 have been excluded from the expression as is manifest in Eq.(22). It is apparent mathematically that it has the energy-momentum covariance. Rohrlich designated it as a kind of renormalization similar to that in quantum electrodynamics.

If Rohrlich's definition is applied to the condenser in the previous section, what result will be obtained ? Inserting the integrand of Eq.(14) or (15) into (6) or (22) yields

$$R^\mu = \gamma[\,1,\ \boldsymbol{\beta}\,]\,(\varepsilon_0 E_x^2/2)\,ABC$$

in either case of motion parallel with or perpendicular to the field. If we disregard the mechanical aspect of the condenser and consider only the energy of electromagnetic field, then it can be said we had the same result as deduced in the previous sections.

5.2 Model with ideal gas

The theory presented hitherto is applicable not only to the electromagnetic field but also to another type of field.

Let us take a tangible example of perfect fluid [12], whose energy tensor is given by

$$T_g^{\mu\nu} = \begin{bmatrix} W_0 & \mathbf{o} \\ \mathbf{o} & \mathsf{I}\, p \end{bmatrix},$$

where W_0 and p are the energy density and pressure of the fluid, respectively. The Lorentz transformation of the tensor above is given by [21]

$$\Lambda\, [T_g^{\mu\nu}]\, \Lambda^t = \begin{bmatrix} \gamma^2(W_0 + \beta^2 p) & \gamma^2(W_0 + p)\boldsymbol{v}/c \\ \gamma^2(W_0 + p)\boldsymbol{v}/c & \gamma^2(W_0 + p)\boldsymbol{v}\boldsymbol{v}/c^2 + \mathsf{I}\, p \end{bmatrix}. \tag{23}$$

The space-time component has an extraneous momentum $\gamma^2 p\boldsymbol{v}/c^2$ due to pressure p, in addition to that from mass density W_0/c^2. Accordingly, the momentum of perfect fluid does not transform covariantly with energy, so long as the pressure is not equal to zero.

Next, consider a case where ideal gas is confined in a balloon. The pressure of the gas maintains balance with the tension of the envelope of the balloon. Utilization of the spherical symmetry of the balloon simplifies the derivation of the stress tensor of the envelope.

The total force which one of the hemispheres receives from the ideal gas is equal to the pressure multiplied by the area of the great circle $p\, \pi R^2$, where R is the radius of the balloon. This force balances that of the tension of the envelope multiplied by the circumference of the great circle $\int \tau\, \delta(r - R)\, \mathrm{d}r \times 2\pi R$, where $\tau\, \delta(r - R)$ is the strength of tension of the skin of the envelop. The thickness of the envelope has been idealized to be infinitesimally thin by employing the Delta function. This argument follows

$$pR = 2 \int \tau\, \delta(r - R)\mathrm{d}r.$$

With this tension $\tau\, \delta(r - R)$, the stress tensor of the envelope is expressed as [13]

$$\mathsf{T}_b = \tau\, \delta(r - R)\, (\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}), \tag{24}$$

[12] The property of fluid is quite different from that of electromagnetic field. However, it can be treated in a similar manner, so long as the field is represented by an energy-momentum tensor.

[13] The contribution of the $\hat{r}\hat{r}$ component becomes negligibly small compared with the $\hat{\theta}\hat{\theta}$ or $\hat{\phi}\hat{\phi}$ component, where the thickness of the envelope tends to 0. In other words, the 3-dimensional stress tensor degenerates into 2-dimensional.

where a spherical co-ordinate system has been employed.

The total momentum contributed by the mechanical stresses in the envelope of the balloon is found by the integration over space

$$\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \int_0^R r^2\gamma^{-1}\mathrm{d}r \; (-\gamma^2 \mathsf{L}\cdot\mathsf{T}_b\cdot\boldsymbol{\beta}/c) \quad = -\gamma\frac{pV}{c^2}\boldsymbol{v}, \tag{25}$$

where the z axis has been selected to coincide with the direction of the velocity of the balloon ($\boldsymbol{v} = \hat{z}v$). The $V = 4\pi R^3/3$ stands for the volume of the balloon. The backward momentum (25) within the envelope skin cancels exactly the extraneous momentum of the ideal gas in Eq.(23) when integrated, so that the system of ideal gas and balloon in union transforms covariantly as a four-vector.

Depicted physically, energy flows forwards in the fluid within the balloon and backwards in the envelope outside. The direction of energy circulation is opposite to the case of the electron.

Application of Rohrlich's vector to this case gives

$$R^\mu = \gamma[\,1,\boldsymbol{\beta}\,]\;\; W_0 4\pi R^3/3,$$

which is the same with the final result as attained above. But, this vector ignores entirely the mechanism of confinement of the ideal gas.

6 Complement

6.1 Momentum factor

A part of field is generally affected by the other parts of the field, so that the physical behavior of that part cannot be specified independently of the other parts. The magnitude of momentum of one part assumes a different value from what it would have if it were alone. The increment (or decrement) of momentum due to the interaction with the adjacent part viz. stresses is given by $-\gamma^2\mathsf{L}\cdot\mathsf{M}_0\cdot\boldsymbol{\beta}/c$. The incremental momentum is not necessarily along the direction of the velocity of the system.

If we introduce the term of "momentum factor" to be the ratio of the space-time components to the time component in the rest frame multiplied by $\boldsymbol{\beta}$ in Eq.(12), it varies dependent on the circumstances. In the case of electrostatic field, it has the maximum value of 2 for vertical motion to the field, while the minimum value of 0 for parallel motion. The factor for the cavity in the previous section took the value of $1 + k_z^2/k^2$, which is between 0 and 2 since $k_z^2 < k^2$. The factor 4/3 for the electron comes from the specialty of the Coulomb field having spherical symmetry.

The factor for ideal gas is found to be given by $1 + p/W_0$ from Eq.(23), which changes with pressure.

Incidentally, this rule for ideal gas is applicable to the case of electromagnetic field, obtaining 0 for the parallel motion with insertion of $p = -\varepsilon_0 E_x^2/2$, and 2 for the vertical motion with $p = \varepsilon_0 E_x^2/2$, where $W_0 = \varepsilon_0 E_x^2/2$. For the spherically symmetric field, putting $p = (1/3)\varepsilon_0 \overline{E^2}/2$ and $W_0 = \varepsilon_0 \overline{E^2}/2$, we have $1 + 1/3 = 4/3$.

6.2 Relativistic invariance as a tensor

The electromagnetic field is a physical entity which is represented by the field tensor $F^{\mu\nu}$ [14] and the energy-momentum tensor (1), which are invariant under the Lorentz transformation. Without exception this law is applicable to the Coulomb field surrounding a charged body. The electromagnetic energy and momentum in general do not need to constitute a four-vector. In other words, the electromagnetic field is always relativistically invariant in terms of tensors, the energy and momentum being only a part (the first column or row) of the energy tensor $T^{\mu\nu}$. The invariance of the field as a tensor is irrelevant to whether the energy and momentum constitute a four-vector or not.

If a system is closed by some means so that stresses are cancelled or removed [$\mathsf{M}_0 \to 0$ in Eq.(12)], then the system is invariant both as a tensor and as a four-vector. Conversely, if the system does not form a four-vector, it is not closed, having stresses caused by the interaction with exterior. Recall that the electrostatic field can not exist by itself without any supporter, as is demanded by Earnshaw's theorem.

6.3 Charged particles and bodies — Poincaré's forces versus Rohrlich's renormalization —

In this paper, the radius r_q of the spherically charged body was not specified ($r_q = r_e$ in the special case of the electron), so that the theory presented here can be applied to an arbitrary radius. This arbitrariness comes from the fact that Maxwell's equations or classical electromagnetics do not furnish any information on the magnitude of a unit charge.

[14] $F^{\mu\nu} = \begin{bmatrix} 0 & -\boldsymbol{E}/c \\ \boldsymbol{E}/c & -\mathsf{B} \end{bmatrix}$, where B is the associated tensor of $\boldsymbol{B}$.

Apart from the magnitude of charge, the characteristic of the Coulomb field of an electron is quite analogous to that of a spherically charged body. Since there is forward energy flow in the outer region as depicted in Fig. 2, there must be backward energy flow in the inner region to satisfy the law of energy conservation. In the interior region, therefore, there must be tension (negative pressure), viz. cohesive forces of some kind. The kind of such forces is known in case of charged objects other than elementary particles. For elementary particles, we should not miss the possibility of gravitational force, though not decisive.

In passing, a salient contrast of the Poincaré model with Rohrlich's treatment will be pointed out. In the former model, the stresses of the field are cancelled with those of non-electromagnetic origin. On the other hand, the latter treatment mathematically removes the electromagnetic stresses, and hence it ignores the flow of energy due to stresses. Therefore, it produces the same result taking a short-cut. The definition of Rohrlich's vector is thus artificial or phenomenological.

6.4 Logical coherence

Although various proposals of amendments or re-definitions of physical quantities have been offered so far to solve the questions associated with the electromagnetic mass, this paper revealed a self-consistent physical picture based on the conventionally (unanimously) approved physical laws.

In this paper, uniformly moving systems of electromagnetic field and matter were treated in the sense that every constituent part of the system moves at a constant speed. It may be supposed therefore that every constituent part should make a linear motion in parallel with each other, so that the circulation of energy may not occur. One may otherwise regard it as a paradox that energy circulation takes place in a uniformly moving system, while everything stands still when the system is at rest. However, *the phenomenon that translation of stresses causes energy flow is an immediate consequence of the definition of energy itself*: (energy) = (force) $\times$ (distance) applied per unit time. This is a reflection of the logical coherency of the natural law.

It is important to remark here that the energy circulation within a system has no direct effect to the external world, provided that the system is closed. This property adduces the physical foundation of possibility for the renormalization as Rohrlich exhibited.

6.5 Trouton and Noble's experiment

The theory in this paper makes it easy to rigorously understand Trouton and Noble's experiment of fame [22]. This experiment corresponds to the situation where a charged condenser moves in an oblique direction with respect to the plates of the condenser. In this oblique movement, it was supposed that turning couple might be generated, since the positive and negative charges on the condenser receive forces in opposite directions perpendicular to the velocity because of the magnetic fields produced by the other charges. This turning couple is calculated to be

$$\boldsymbol{N}_e = -\hat{z}A(QE_x/2)\beta^2 \cos\varphi \sin\varphi$$

according to elementary electromagnetics, where Q is the charge on the electrode BC in Fig.1, and φ is the angle between the direction of the velocity and the electric field E_x in the $x-y$ plain.

As the charges are fixed on the electrodes, this turning couple is transmitted through the law of action and reaction to the condenser. The induced turning couple is computed from the momentum occurring in the separator of the condenser in terms of the stresses in it. As there is pressure p equal to $\varepsilon_0 E_x^2/2$ in the separator, there is an amount of energy flow

$$\int (-\gamma^2 \mathsf{L} \cdot \mathsf{T}_p \cdot \boldsymbol{v}\ )\, \mathrm{d}^3x$$

through the separator, where $\mathsf{T}_p = -p\hat{x}\hat{x}$. The cross product of the position vector $\boldsymbol{r}$ with the above momentum equals the angular momentum. The turning couple or torque is the time derivative of this quantity, yielding

$$\boldsymbol{N}_p = A\,BCp\,\boldsymbol{\beta}\cdot\hat{x}\ \ \boldsymbol{\beta}\times\hat{x} = -\hat{z}A\,BCp\,\beta^2\cos\varphi\sin\varphi,$$

where $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$. The turning couple $\boldsymbol{N}_p$ is equal to $\boldsymbol{N}_e$, since $BCp = QE_x/2$, where $Q = BC\varepsilon_0 E_x$. This mechanical turning couple counterbalances that generated by the Maxwell's stresses M_0 in accordance with the stability condition (10) as a coupled system of charge and matter. Another description is found in Pauli [23], which makes no mention of the electromagnetic stresses necessary for energy and momentum conservation.

To return from the digression, it is readily confirmed that our systems in question also do not rotate as a whole, although energy circulation occurs in it. This is because the cross product of the position vector with the total momentum due to stresses (19) vanishes always. The energy circulates so that the system has no angular momentum.

7 Reflections

The essentials of the solution of the problem on the electromagnetic mass has been elucidated. However, there remains tasks to resolve the conflicts or confusions associated with this problem, since the controversies have persisted for a long time.

Among a variety of opinions so far to interpret the electromagnetic mass, the renormalization procedure introduced by Rohrlich [9] appears to be most widely disseminated. If we accept Rohrlich's point of view without discretion, we may be put to bewilderment. This is because his point of view is essentially incompatible with the authentic laws of physics.

The following is a discourse how these confusions are resolved, based on the new finding that stresses produce momentum without mass.

7.1 Lorentz-invariant four-vector

It is known that there exists a four-vector

$$cP^{\mu} = \int T^{\mu\nu} d\sigma_{\nu} \tag{26}$$

which is independent of surface σ_{ν}, provided that the tensor is bounded or satisfies $\partial T^{\mu\nu}/\partial x_{\nu} = 0$. For a charged particle, however, the divergence of electromagnetic energy-momentum tensor does not vanish, since $\partial T_e^{\mu\nu}/\partial x^{\nu} = -F^{\mu\nu} j_{\nu}$. In this context, Rohrlich found that cP^{μ} of Eq.(26) can be made Lorentz invariant, if one chooses the surface $d\sigma^{\nu} = n^{\nu} d\sigma$ ($n^{\nu} = v^{\nu}/c$) [8]. He defined it ($P^{\mu} = R^{\mu}/c$) to be the momentum of the electron, calling n^{ν} a time-like vector normal to the hyper-surface element $d\sigma$. Besides this geometrical explanation, he and his supporters did not give the physical reason why such a choice of surface brings forth a Lorentz-invariant vector.

To find the physical meaning of it, we note that the energy tensor $T^{\mu\nu} = T_e^{\mu\nu}$ in Eq.(26) is the Lorentz transformation of the tensor $T_0^{\mu\nu}$ in the rest frame of the electron as is given by Eq.(4). The normal vector $n^{\nu} = (1, \boldsymbol{\beta})$ or its covariant component is written

$$n_{\nu} = \Lambda_{\nu}{}^{\lambda}(-\boldsymbol{\beta})\, n_{\lambda}^{(0)}, \tag{27}$$

where $\Lambda(-\boldsymbol{\beta})$ represents the inverse of the Lorentz transformation (3) with $\boldsymbol{\beta}$ replaced by $-\boldsymbol{\beta}$, and where $n_{\lambda}^{(0)} = (1, 0, 0, 0)$ is the normal vector in the rest system of the electron. With $\Lambda_{\nu'}{}^{\nu}(\boldsymbol{\beta})\Lambda_{\nu}{}^{\lambda}(-\boldsymbol{\beta}) = \delta_{\nu'}{}^{\lambda}$, Eq.(26) assumes the form

$$cP^{\mu} = \int \Lambda^{\mu}{}_{\mu'} T_0^{\mu'\nu'} n_{\nu'}^{(0)} d^3x_0.$$

Next, the normal vector is split into the time and the space parts: $n_\nu^{(0)} = (1,\ 0_n)$, where 0_n is the 3-dimensional null vector ($n = 1, 2, 3$). As $T_0^{\mu\nu}$ is given by Eq.(11), the above equation is written

$$cP^\mu = \Lambda^\mu{}_{\mu'} \int (T_0^{00}, T_0^{m'n} 0_n) d^3x_0, \qquad \mu = 0, 1, 2, 3 \quad \text{and} \quad m, n = 1, 2, 3, \tag{28}$$

where the space-time components $T_0^{0n} = T_0^{m0} = 0$ have dropped out.

The quantity $T_0^{00} d^3x_0$ in Eq.(28) represents electromagnetic energy included in an infinitesimal volume $d^3x_0 = dx_0 dy_0 dz_0$ in the rest frame of the electron. It is seen that the spatial component $T_0^{m'n} 0_n d^3x_0 = 0^{m'}$ contributes noting to the momentum P^μ, since 0_n is the spatial null vector. This fact implies that the impulse due to Maxwell's stresses $-T_0^{m'n} = \mathsf{M}_0$ acting on the plane $n^\nu = (1, n^n)$ is neutralized with that on the other side $(1, -n^n)$. This is because the plane n^ν is moving with velocity $v^\nu = n^\nu c$, or the plane $n_\nu^{(0)}$ is at a standstill relative to the Coulomb field around the electron.

Under this condition, the energy $\int T_0^{00} d^3x_0$ is in uniform motion with velocity $v^\mu = (c, v^m)$, so that it is natural to expected that there should be energy flow given by $\int T_0^{00} d^3x_0\ v^\mu$. Mathematically, the Lorentz transformation of the integral $\int T_0^{00} d^3x_0(1, 0^{m'})$ in Eq.(28) yields the momentum $P^\mu = \gamma m_e v^\mu$, where $m_e = \int T_0^{00} d^3x_0/c^2$. This is the value that Rohrlich expected. In this connection, we must bear in mind that we are sitting in the reference frame moving with the electron to see the stress components $T_0^{m'n}$, so that we do not observe momentum (energy flow) due to Maxwell's stresses as in Fig.2, because the impulses acting on both sides of a plane normal to v^m cancel each other. This never means that Maxwell's stresses have died away by the renormalization, as Rohrlich may have conceived so.

7.2 Argument

In his Comment on Boyer's objection [10], Rohrlich argued in the following fashion [11]. In the Abraham-Lorentz-Poincaré model, the total energy tensor

$$T^{\mu\nu} = T_e^{\mu\nu} + T_p^{\mu\nu}$$

satisfies the stability condition $\partial T^{\mu\nu}/\partial x_\nu = 0$, where $T_e^{\mu\nu}$ and $T_p^{\mu\nu}$ denote electromagnetic and non-electromagnetic components, respectively. Then, there is a four-vector

$$cP^\mu = \int T_e^{\mu\nu} d\sigma_\nu + \int T_p^{\mu\nu} d\sigma_\nu \equiv cP_e^\mu + cP_p^\mu, \tag{29}$$

which is independent of surface σ_ν, but each component P_e^μ or P_p^μ is not separately independent of σ_ν. Selecting the two surfaces, (I) inertial system of the observer and (II) rest system of the particle, he calculated the contributions to the momentum P^μ from electromagnetic and mechanical parts.

In the first case (I), his calculation showed that the momentum contribution from the electromagnetic part is the same as the Poynting vector gives, and he remarked that P_e^μ is not a four-vector. From the discussions associated with Eq.(17), the physical reason why he has obtained this result may be clear. In this case the frame of the observer runs with the velocity $-v^m$ relative to the Coulomb field of the electron, so that there is additional energy flow through the plane perpendicular to v^m being caused by Maxwell's stresses, since the magnitude of impulse acting on the plane n^ν is different between the front and the rear sides of the plane. This energy flow $-\gamma \int M_0^{mn} v_n d^3x_0$ has caused the discrepancy of momentum from the expected value of the direct energy transportation $\gamma \int W_0 d^3x_0 v^m$.

In the second case (II), Rohrlich presented a detailed computation obtaining a Lorentz-invariant vector. The matrix calculus in section 5 gave the same result in a generalized way, implying that *any physical system represented by a tensor $T^{\mu\nu}$ can be made Lorentz-invariant* or "renormalized" with the normal (27). This effect may be ascertained using his expression (29). The left-hand side is always Lorentz-invariant by the assumption of closed structure, and the first term cP_e^μ in the right-hand side is made invariant by selecting n_ν as Eq.(27). On this condition, the last term $cP_p^\mu = cP^\mu - cP_e^\mu$ must be invariant, too. This aftermath implies that the mechanical part is simultaneously "renormalized". If the choice of n_ν (27) causes the self-stress of the electron to vanish in reality as he claimed, then the self-stress of the "matter" to oppose the former has to vanish as well. This is an obvious contradiction.

Along with the physical description of the previous subsection, it will now be understood that the magnitudes of the spatial components n^n of the normal n^ν do nothing but mathematically adjust the contribution of the stresses to momentum, which becomes zero when $n^n = v^n/c$ or $n_n^{(0)} = 0_n$.

7.3 Electron as a closed system

Let us delve a little more into the structure of the electron. Whatever model of the electron we may conceive, the Coulomb field around it is expressed by Eq.(21) with q replaced by $-e$, as long as the charge distribution is spherically symmetric. When an electron carrying the Coulomb field moves at a constant speed, there is energy current of two types, direct transportation of (electromagnetic) energy and translational movement of (Maxwell's) stresses, as elucidated in section 4. This phenomenon of energy flow is independent of the kind of electron models, as far as the Coulomb field is concerned. Therefore, even in the treatment by Rohrlich, there must be the same energy current as in Fig.2 [15]. Then, from the law of energy conservation, this energy current must penetrate through the middle part of the electron in the backward direction. The backward energy flow implies that there must exist tensional stresses in the inner part of the electron irrespective of electron models.

This fact can be confirmed more vividly by the investigation to follow. For any closed system, we have

$$\int T_0^{mn} d^3x_0 = 0, \tag{30}$$

applying Gauss's theorem to $\partial T_0^{\mu\nu}/\partial x_\nu$ [14], [18]. The volume integral extends over all the system. This equation states that the total stresses in a closed system should vanish as a whole to maintain its equilibrium. For an electron lying in the rest system, the domain of integration (30) is divided into two, inner ($r < r_x$) and outer ($r_x \leq r$) regions :

$$\int_0^{r_x} T_0^{mn} r^2 d\Omega dr + \int_{r_x}^{\infty} T_0^{mn} r^2 d\Omega dr = 0.$$

As far as the radius r_x is sufficiently large compared with the classical radius r_e, the stress tensor of an electron is given by $-T_0^{mn} = \mathsf{M}_0 = \varepsilon_0 \boldsymbol{E}_0 \boldsymbol{E}_0 - \varepsilon_0 \boldsymbol{E}_0^2/2$ with $\boldsymbol{E}_0$ of Eq.(21). The integration outside r_x is evaluated to be [16]

$$-\int_{r_x}^{\infty} T_0^{mn} 4\pi r^2 dr = -\frac{1}{3} \cdot \frac{e^2}{8\pi\varepsilon_0 r_x} \cdot \delta^{mn}, \tag{31}$$

while the remainder of the stresses inside r_x becomes

$$-\int_0^{r_x} T_0^{mn} 4\pi r^2 dr = \frac{1}{3} \cdot \frac{e^2}{8\pi\varepsilon_0 r_x} \cdot \delta^{mn}. \tag{32}$$

This deduction signifies that the Maxwell's stresses existing outside an arbitrary spherical plane of r_x has to be compensated for by the stresses of the interior region, so long as the Coulomb field surrounding the electron to keep its shape. The existence of the interior stresses (32) is vitally important to balance the exterior stresses of the Coulomb field (31).

[15] As the definition of the electric field (force acting on a unit test charge) is same for any field (regardless of the Coulomb or radiation field), Maxwell's stresses cannot be neglected while we are dealing with electromagnetic field at all.

[16] $\mathsf{M}_0 = (1/2)\varepsilon_0 \boldsymbol{E}_0^2 (2\hat{r}\hat{r} - 1)$.

This is physically ascertained from the observation that there is electric field penetrating through the spherical plane $r = r_x$. More precisely, the outer region exerts tensile force on the inner region through the spherical plane $r = r_x$ by dint of electric field perpendicular to the plane, and the inner part exerts attractive force on the outer part by the law of action and reaction. Owing to this attractive force, the outer part of the Coulomb field is prevented to fly apart. If the inner part is removed, for instance, by a quick motion of the electron, the outer part of the field will go away to become radiation field (note that there exists no electrostatic field without support). With this mechanism, the Coulomb field of the electron has a dual character, viz. dependency on the charge and independency from it, intermediated by Maxwell's stresses. From the explication above, it may be certain that in the inner part of the electron there must be something that will compensate for the stresses of the outer part. For the Poincaré model, it was a set of cohesive forces.

The simplest model is the charge distribution on the spherical surface at the classical electron radius r_e. Substitution of $r_x = r_e$ in Eq.(32) yields

$$\int_0^{r_e} T_0^{mn} 4\pi r^2 dr = \frac{1}{3} W_0 \delta^{mn}.$$

This is nothing but the forces required for the Poincaré model. It should be remarked here that the Poincaré stresses have been deduced without resort to charges.

In other models conceivable, however, the electromagnetic and non-electromagnetic domains may be intermixed with each other. Even in such models, the contribution of electromagnetic field to the total stresses of the system is determined by Maxwell's stress tensor [17], and they must be compensated for by non-electromagnetic stresses in order that the electron system maintains its equilibrium. Since the integrated values of Maxwell's stress tensor of the spherically symmetric object are negative as is seen from Eq.(31), the compensatory stresses are of positive value [18]. Although we do not know at present what kinds of stresses (forces) they are, they must exist by all means in order for the Coulomb field or static field to survive. Otherwise, why can the field maintain its shape, each part of it being separated in distance away from the charges ?

It is emphasized here that the discussions in this paper have been presented without regard to any particular charges but only in terms of electromagnetic field. For example,

[17]In the innermost of the electron, there is a possibility that Maxwell's equations may not hold accurately. Admitting that it is so, the gist does not change.

[18]Depending on the configuration of the charged object, the compensatory stresses can be of negative nature being different from the Poincaré cohesive forces, as in case of the charged condenser.

Eqs.(31) and (32) hold for every Coulomb field regardless of charges, so long as r_x is at least taken larger than r_e. The necessity of existence of the counter-stresses to the Coulomb field has been deduced from Maxwell's stresses, different from the conventional reasoning by use of charge distribution.

7.4 Density of mass and stresses in field

Normally, the term "electromagnetic mass" is referred to the mass of an electron. It may be reasonable and more convenient, however, to generalize this term so that we can consider the momentum or mass contained in an infinitesimal volume of electromagnetic field. Then, the electromagnetic mass of a charged particle is naturally given by the integration of mass density over the space including the particle.

Now, there may be at least three kinds of definition of mass: gravitational mass, inertial mass (momentum divided by velocity) and relativistic definition $m = W_0/c^2$. Needless to say, they are equally applicable to matter (a rigid body), and to an infinitesimal volume of fluid. However, *care must be taken when we apply the second (inertial) definition to electromagnetic field.* This is because the momentum of electromagnetic field generally involves the contribution from Maxwell's stresses. To be rigorous, even in fluid, there is contribution from stresses which is normally negligibly small.

The electromagnetic momentum contained in a unit volume is given by the Poynting vector (17) or by the space-time component of the energy-momentum tensor (12). We write here the momentum density of field as

$$\boldsymbol{g} = \boldsymbol{E} \times \boldsymbol{H}/c^2 = \gamma^2(W_0\boldsymbol{\beta} - \mathsf{L} \cdot \mathsf{T}_0 \cdot \boldsymbol{\beta})/c = \gamma^2(W_0/c^2)(\mathsf{I} - \mathsf{L} \cdot \mathsf{T}_0/W_0) \cdot \boldsymbol{v}, \qquad (33)$$

where T_0 represents the stress tensor of field in the rest frame. The quantity appeared in the equation above

$$\mathsf{F} = \mathsf{I} - \mathsf{L} \cdot \mathsf{T}_0/W_0 \qquad (34)$$

is the mathematical formulation of the momentum factor mentioned in the previous section. Generally, the momentum factor is a non-dimensional tensor quantity, because the the momentum due to stresses is not always in parallel with the motion of field. If there were no stresses $\mathsf{T}_0 = 0$ in the field, the momentum factor F would become unity or the unit tensor, so that we would have

$$\boldsymbol{g} = \gamma^2 W_0 \boldsymbol{v}/c^2. \qquad (35)$$

Since field in general has stresses in it, the magnitude of momentum density is modified as

$$\boldsymbol{g} = \mathsf{F} \cdot \gamma^2 W_0 \boldsymbol{v}/c^2 = \mathsf{F} \cdot \gamma^2 m\boldsymbol{v}, \tag{36}$$

where $m = W_0/c^2$ is the density of mass in field.

For electrostatic field $\mathsf{T}_0 = \mathsf{M}_0 = \varepsilon_0 \boldsymbol{E}_0 \boldsymbol{E}_0 - \varepsilon_0 \boldsymbol{E}_0^2/2$, the factor is computed as

$$\mathsf{F}_e = 2\,\mathsf{I} - 2\,\mathsf{L} \cdot \boldsymbol{E}_0 \boldsymbol{E}_0/\boldsymbol{E}_0^2 \,. \tag{37}$$

It is easy to confirm that this factor F_e varies from 0 for $\boldsymbol{E}_0 \parallel \boldsymbol{\beta}$ to 2 for $\boldsymbol{E}_0 \perp \boldsymbol{\beta}$, as was already seen in the condenser model of section 3.

For the ideal gas with $\mathsf{T}_0 = \mathsf{I}\,p$, the momentum factor assumes $\mathsf{F}_g = 1 + p/W_0$ [19].

Since the momentum of matter (or a mechanical body) is defined to be the mass times velocity , so the mass is found from the momentum divided by the velocity. This method of determination of mass is not directly applicable to field, because the momentum factor F is not unity in Eq.(36). Therefore, in order to find out the mass density in field from momentum density, it is necessary to reduce the momentum factor F to unity. That is to say, the mass density m in field is given by the coefficient of

$$\mathsf{F}^{-1} \cdot \boldsymbol{g} = \gamma^2 m\boldsymbol{v}$$

with respect to $\gamma^2\boldsymbol{v}$. The F^{-1} denotes the inverse of F. It may be clear that the above value is also equal to the first term of Eq.(33) discarding the stress term. With this treatment, it is always possible to find the mass density contained in (electromagnetic) field.

While the distribution of mass density of the Coulomb field of a charged particle is spherically symmetric, the momentum density is larger in the tropical region and smaller near the poles, as depicted in Fig.2. In this fashion, each part of the Coulomb field has

[19] From the state equation of gas $pV = nRT$, the correction factor due to extraneous momentum $\mathsf{F}_g - 1$ for a gas is computed to be

$$p/W_0 = p/\rho c^2 = RT/Mc^2,$$

where ρ, R and M are the mass density, gas constant and molecular weight of the gas, respectively. For the air on the surface of the earth, this value is of order of $p/\rho c^2 = 10^{-15}$, which is totally negligible compared with unity.

In cosmology it is known that the factor for a star is approximated as

$$p/\rho c^2 \approx r_g/r,$$

where r_g and r are the Schwarzschild radius and distance from the center of the star, respectively. It is estimated that this factor reaches 10^{-4} for white dwarfs and near unity for neutron stars.

different momentum density, causing the energy circulation. For the electron, the *averaged momentum factor* is calculated to be

$$\int_{r>r_e} \mathsf{F}_e d^3x \Bigg/ \int_{r>r_e} d^3x = \frac{4}{3},$$

so that we have the value

$$\int_{r>r_e} \mathsf{F}_e^{-1} \cdot \boldsymbol{g}\, d^3x \;/\; \gamma \boldsymbol{v} = m_e$$

for the electromagnetic mass. Thus, the integration of the mass density (excluding the stress term) over the space gives us exactly the expected value of electromagnetic mass.

As was elucidated by the analytical treatment above, stresses have nothing to do with mass (see footnote 9), although it causes energy current and hence momentum. The reason why the energy current due to stresses accompanies momentum is that the stresses give impulse (force during the time concerned) onto the plane perpendicular to the current. So, the existence of stress in field implies that the field is unstable in itself, unless the stress is counterbalanced by any other means such as a charge, electrodes or an envelope of a balloon. Earnshaw's theorem and the diffusion effect of gas are examples of the unstable character of field. Conversely, if stresses are compensated by some means, the system acquires stability to become a set of matter like a charge, a condenser or a balloon.

Matter or a mechanical body is therefore stable in itself, because the stresses are balanced within it. Stress, however, can not exist without any sustainer of them or medium called field. While the stresses T_0 have no mass by themselves, the field as their sustainer should have mass W_0/c^2. In case of electromagnetic field, the mass comes from energy density $\varepsilon_0 \boldsymbol{E}_0^2/2$ and $\boldsymbol{B}_0^2/2\mu_0$. The mass of gas from the particles and the kinetic energy of them.

To be more inquisitive, the charge itself has no mass [20]. There is a possibility, however, that the mass of the "matter" to furnish the counter-stresses is not zero. In such a case, non-electromagnetic mass is simply added to the electromagnetic mass similar to m_p in Eq.(5). The investigation of the mechanism of the counter-stresses, that is, whether the electron contains non-electromagnetic mass or not, is a theme of study in future.

7.5 Feasibility of renormalization

As mentioned previously, Rohrlich claimed to the effect that the self-stress of the electron vanishes by his definition of the covariant vector, or by means of renormalization. Moreover,

[20] According to Maxwell's equations, the charge is nothing but the source or sink of electric field. If the charge has mass, it must have come from some other sources.

there seems to prevail a notion that the renormalization is the ultimate solution to the charged particles. As was illustrated in section 5, the renormalization technique or cutoff procedure can be equally applied to charged condensers and even to a balloon. In such cases, what does the "renormalization" mean ? Does the envelope of the balloon to confine the fluid become unnecessary, if the "renormalization" has been performed ? Nevertheless, is he correct to say that Poincaré's forces and the factor 4/3 are incorrect ?

Or, are both views, Rohrlich's and Poincaré's, correct representing two aspects of our approach to physics, as Campos and Jiménez suggested [12] ? It must be noted here that the two views are contradictory from each other, as Butler remarked [25]. He called for a revision of the Poynting vector for charged objects according to Rohrlich's definition, explaining that it removes some "paradoxes" where charges are involved. Though his mathematical treatment is a little different, the essence is to remove the stress term from the Poynting vector. This is equivalent to assume that the electromagnetic energy current consists only of the transportation of energy, so that the physical picture is made extremely simple. As a reverse effect, this sort of simplification incurs other "paradoxes" such that it represents no more the power flow in a transmission line. Rohrlich's definition thus leads to a different electromagnetic momentum, as Boyer argued that it destroys the conceptual simplicity of classical electrodynamics [10]. The two views thus cannot consist with each other. *How is this antinomy solved ?*

The nature of the electron and its surrounding Coulomb field can never be changed by means of our definition or theory. Maxwell's stresses are the basis of the electromagnetic forces which cannot be negated. Hence, the vanishment of self-stress of the electron is no more than the outward appearance brought forth by the renormalization technique [21], in the sense that the self-stress within the closed system of the electron exercises no net influence on the outside world. The energy circulation within a charged object producing no net angular momentum illustrated in section 6 is an example of this characteristic of a closed system.

Stated conversely, this very property of the closed system made the renormalization possible, since the inner stresses can be separated from the external world. Thus, the renormalization cannot be considered as the final solution to the problem concerning the structure

[21] Rohrlich seems unaware of the integral condition (10) or (30) for a closed system. For example, he wrote "$T^{(0)}_{\mu\nu} = 0$ otherwise than $\mu = \nu = 0$" in Eq.(6-14) of his book [9] for the Abraham-Lorentz-Poincaré model. The vanishment of divergence $\partial_\mu T^{\mu\nu} = 0$ in Eq.(6-11) does not necessarily mean that the stresses $T^{(0)}_{mn}$ themselves also have vanished entirely, although the space integration of them do as Eq.(30).

of the electron or a charged particle. It is practically convenient simplifying the treatment of electrons interacting with the external world.

This is analogous to a mechanical system which consists of a number of particles bounded by a closed surface. The internal forces or interaction among the particles and the boundary do not make a net contribution to the external world. Based on this condition, the closed system behaves as a single (rigid) body ignoring the internal structure. However, in order to know the mechanism inside the body, the internal forces are indispensable. The distinction of the mechanical system from the electron is the contrast that the boundary between the internal and external domains is clear-cut in the mechanical case as of a balloon, whereas the boundary is spatially crossbred in the electron.

7.6 Epitome

Although the problem of electromagnetic mass has been complicated in the course of history so that the detailed elucidations or discussions have been required to resolve the confusions, the answer is epitomized rather simply.

The energy current or momentum in field consists of the two types, viz. transportation of energy and translation of stresses, so that the extraneous momentum should be subtracted from the total momentum in order to educe the intrinsic energy or mass involved in it. The integration of the mass density over the space including the electron gives automatically the electromagnetic mass as is expected, without the discrepancy.

As a corollary, if the system is closed, the integration of the extra momentum density over the entire system vanishes, so that the extraneous momentum inside the closed system produces no net effect on the outside world. This very nature of a closed system made the useful renormalization possible. The reality is not that the artificial renormalization theory actually has removed Maxwell's stresses.

The inadvertent application of the second definition of mass (momentum divided by velocity) to electromagnetic field has been the central cause of the confusions concerning the electromagnetic mass. We should not mistake that the extra momentum (1/3) has been brought forth also from mass. The vanishment of the extra momentum in a closed system seems to have amplified the confusions, as the conventional arguments have been made only in terms of integrated quantities over the system rather than from the analytical point of view. It is remarked here that, for the integrated quantity over a closed system, the second definition of mass is directly applicable, since the momentum factor reduces to unity with stresses being neutralized.

There may be still left not a few confusions which could not be touched here. Hopefully they will be resolved in accordance with the explanation presented in this paper.

8 Conclusions

So far the problem of the electromagnetic mass has been pursued to obtain a covariant behavior of the electron. The more radical issue is why we seemed to have two values of mass for the electron, the covariance being the consequent of its resolution.

On the basis of a wide-range physical consideration rather than the conventional treatment in consequentialism, this paper made an analytical examination into the definition of mass itself, and deduced the fundamental fact that the energy flow or momentum in field is composed of two types. Especially in case of electromagnetic field, the effect of this composite structure of momentum is not negligible. This characteristic of the field was the primary reason why we seemed to have two values for the electromagnetic mass. The gist of the present paper has been summarized in the end of the last section.

In short, the discrete application of the inertial definition of mass excluding the stress term naturally agrees with the relativistic definition ($m = W/c^2$) irrespective of field or matter. Being applied to the electron, the integration gives the definite value of electromagnetic mass, without the discrepancy. Whether or not the electron possesses additional non-electromagnetic mass is a theme of study in future. The renormalization which is convenient for application cannot be regarded as the real solution, having nothing to do with the internal structure of the electron.

It may be said finally that the essence of the problem on the electromagnetic mass has been clarified to the possible limit of classical field theory, excepting the question what kinds of stresses counterbalance Maxwell's stresses. The answer of this question may be beyond the scope of classical theory. In quantum mechanics, however, the electron is treated as a point charge, having the difficulty of divergence. To attain a real solution, we should note that the electron must have an extension. The evidence that the renormalization is not the ultimate solution will encourage us to investigate further into the source of the unknown stresses or the internal mechanism of charged particles. Interesting researches are being continued to seek for the relevance to gravitational force [26], although our knowledge at present is still limited to reach a definitive solution, as Einstein commented.

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