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Optimal Semilinear Credibility

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1. Introduction

We consider the homogeneous model of credibility theory defined by a sequence

$$\Theta, X_1, X_2, \ldots \tag{1}$$

of random variables, where for $\Theta = \theta$ fixed, the variables $X_1, X_2, \ldots$ are conditionally independent and equidistributed. The structure variable Θ may be interpreted as the parameter of a contract chosen at random in a portfolio, the variable X_i $(i = 1, 2, \ldots)$ as the total cost of the claims of the year i of that contract chosen at random.

The *general credibility premium* of the year $t+1$ is the conditional mean value

$$E(X_{t+1}/X_1, \ldots, X_t). \tag{2}$$

It is the variable $f(X_1, \ldots, X_t)$ closest to X_{t+1} in the least squares sense.

If $f(X_1, \ldots, X_t)$ is assigned to be of the particular linear form

$$a_0 + a_1 X_1 + \ldots + a_t X_t, \tag{3}$$

we obtain the *linear credibility premium* denoted hereafter by

$$\bar{E}(X_{t+1}/X_1, \ldots, X_t). \tag{4}$$

Except for particular cases where (2) and (4) are equal, the general premium is of course closer to X_{t+1} than the linear one. The advantage of the linear premium is that its coefficients $a_0, a_1, \ldots, a_t$ can be estimated statistically.

Premiums with generality and precision between that of (2) and (4) can be considered. In fact, if f is now a real function of one real variable, we define

$$\bar{E}(X_{t+1}/f(X_1), \ldots, f(X_t)) \tag{5}$$

as the best approximation (always in the least squares sense) of X_{t+1} of the particular form

$$a_0 + a_1 f(X_1) + \ldots + a_t f(X_t). \tag{6}$$

The premium (5) may be called a *semilinear credibility premium.* Indeed, (6) is linear in $f(X_1), \ldots, f(X_t)$, but f itself needs not to be linear.
The interesting point is that the advantage of the linear premium, of being statistically estimable, is not lost by the semilinear one. Another advantage of semilinear credibility is that we can give the premium (5) some properties needed for practical reasons. For example, it may be desirable that the premium stays in a pre-assigned interval. This can be achieved when working with a bounded f.
As long as f is fixed, semilinear credibility is in fact a particular case of linear credibility in a multidimensional model as considered in [5], *Jewell.*
The question of the existence of an optimal f, in the sense of minimizing

$$E(X_{t+1} - f(X_1) - \ldots - f(X_t))^2, \tag{7}$$

naturally arises. We take the form of approximation $f(X_1) + \ldots + f(X_t)$, because $a_1 = \ldots = a_t$ for the semilinear credibility premium (6). Then (6) can be written

$$\left(\frac{a_0}{t} + a_1 f(X_1)\right) + \ldots + \left(\frac{a_0}{t} + a_1 f(X_t)\right).$$

But at looking for an optimal f, we let of course vary f. Then the constant term and the coefficients can be dispensed with because they may be considered as being incorporated in f. The question is answered affirmatively in these notes. An optimal f will be denoted by f^* and the corresponding *optimal semilinear credibility premium,* which is then $f^*(X_1) + \ldots + f^*(X_t)$ by

$$E^*(X_{t+1}/X_1, \ldots, X_t). \tag{8}$$

2. Some Results in Hilbert Space Theory

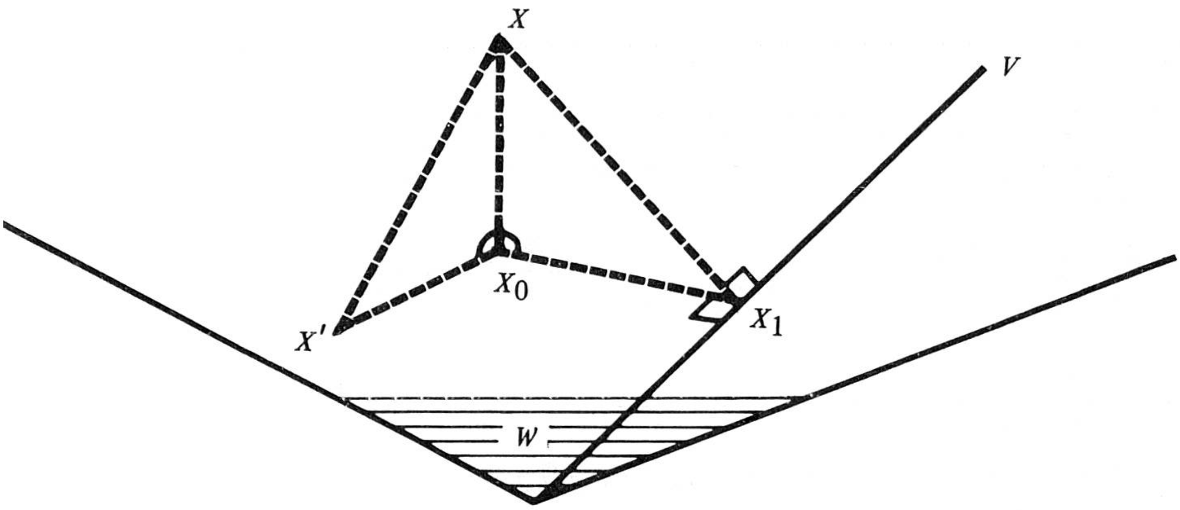
Let H be a *Hilbert* space over the field R of real scalars. In such a space is defined a norm $\| \ \|$ and an orthogonality relation $\perp$ (deriving both from a scalar product). Thus, for each couple X, Y of points in H, the distance $\| X - Y \|$ is defined as a real finite number and the relation $X \perp Y$ is meaningfull (generally false). This norm and orthogonality relation have most of the properties of the same named notions in n-dimensional real Euclidean space. The norm induces a topologie in H. Convergence, closure, ..., are relative to this norm topologie. Some central results in *Hilbert* space theory are the following (see fig. 1). Let W be a closed linear subspace of H and $X \in H$. Then there exists an unique point $X_0 \in W$ such that $X - X_0 \perp W$. By the last relation is meant

that $X - X_0 \perp X'$ for each $X' \in W$. The point X_0 is also the unique point $X' \in W$ making the distance $\| X - X' \|$ minimum. The point X_0 is the orthogonal projection of X upon W and will be denoted by PRO (X/W).
If V is a closed linear subspace of H and $V \subseteq W$, then the projection X_1 of X on V is the same point as the projection of X_0 on V. Thus

$$\mathrm{PRO}(\mathrm{PRO}(X/W)/V) = \mathrm{PRO}(X/V). \tag{9}$$

From a general theorem in the theory of *Banach* spaces results that a finite dimensional linear subspace of a *Hilbert* space is closed.

Fig. 1



3. Hypotheses, Notations and Definitions

To simplifie the language, we shall consider Θ as a real usual random variable. Only a few non essential modifications should be introduced if Θ were multi-dimensional.
The random variables (1) are real finite measurable functions defined on a basic probability space $(\Omega, \mathfrak{I}, P)$. All subsequent random variables are supposed to be defined on this space.
The space L_2 of random variables $X, Y, \ldots$ with finite second order moment is then a *Hilbert* space with norm satisfying $\| X \|^2 = E(X^2)$ and in which the orthogonality relation $X \perp Y$ is equivalent to $E(XY) = 0$. Since the relation $\| X \| = 0$ implies $X = 0$ in a *Hilbert* space, two a.e. (almost everywhere) equal random variables are to be considered as the same point of L_2. If $E(X^2) < \infty$, then X is finite a.e. Nothing essential is changed then if we replace $X(\omega)$ by

zero on the null set $N \in \mathfrak{J}$ where $X(\omega)$ is infinite. Thus we may suppose that the variables of L_2 are finite everywhere.
It is assumed that $X_1, X_2, \ldots \in L_2$.
For $i = 1, 2, \ldots$, we denote by B_i the family of real finite *Baire* functions f of i real variables satisfying

$$f(X_1, \ldots, X_i) \in L_2. \tag{10}$$

Thus f is defined on the real i-dimensional Euclidean space, whereas $f(X_1, \ldots, X_i)$ is defined on Ω.
Hereafter t will be a fixed positive integer. The dependence on t is not always indicated in our notations.
The field of real finite numbers is denoted by R.

4. Some Closed Linear Subspaces of L_2

(At a first reading, the technical demonstrations of 4.1. and 4.3. may be omitted.)

4.1. The set

$$W = \{f(X_1, \ldots, X_t)/f \in B_t\} \tag{11}$$

is a linear subspace of L_2. We prove that it is closed.
Consider the sequence $f_n(X_1, \ldots, X_t) \in W$ converging to $Y \in L_2$. We have to show that $Y \in W$. A subsequence $f_{n_k}(X_1, \ldots, X_t)$ converges to Y a.e. This means that there is a null set $N \in \mathfrak{J}$ such that $f_{n_k}(X_1(\omega), \ldots, X_t(\omega)) \to Y(\omega)$ for every $\omega \in \Omega$ not belonging to N. Define g by $g = \limsup f_{n_k}$. Then, for every ω not belonging to N:

$$Y(\omega) = \lim f_{n_k}(X_1(\omega), \ldots, X_t(\omega)) =$$

$$\limsup f_{n_k}(X_1(\omega), \ldots, X_t(\omega)) = g(X_1(\omega), \ldots, X_t(\omega)).$$

Define f to be equal to g except when g is infinite in which case the value $f(x_1, \ldots, x_t)$ is set equal to zero. Then f is a *Baire* function and $Y = f(X_1, \ldots, X_t)$ a.e. Since $Y \in L_2$, we have $f \in B_t$ and finally $Y \in W$.

4.2. The set

$$V_f = \{a_0 + a_1 f(X_1) + \ldots + a_t f(X_t)/a_0, \ldots, a_t \in R\}, \tag{12}$$

where f is a fixed function in B_1, is a linear subspace of L_2. Since it is finite dimensional, it is closed.
Similarly, the set

$$V'_f = \{a + bf(X_1) + \ldots + bf(X_t)/a, b \in R\} \tag{13}$$

is a closed linear subspace of L_2.

4.3. The set

$$V = \{f(X_1) + \ldots + f(X_t)/f \in B_1\} \tag{14}$$

is a linear subspace of L_2. To prove that it is closed, let $Y_n = f_n(X_1) + \ldots + f_n(X_t)$ be a sequence of points in V converging to $Y \in L_2$. We have to prove that $Y \in V$. The convergent sequence Y_n is a *Cauchy* sequence in L_2. Thus

$$\lim_{n,m} E(Y_n - Y_m)^2 = 0. \tag{15}$$

Setting, for $i = 1, 2, \ldots, t$, $g_{nm}(X_i) = f_n(X_i) - f_m(X_i)$, we have $E(Y_n - Y_m)^2 = E(\sum_i g_{nm}(X_i))^2 = \sum_i E g_{nm}^2(X_i) + 2\sum_{i<j} E(g_{nm}(X_i) g_{nm}(X_j))$. The last sum is non-negative, since for $i \neq j$, $E(g_{nm}(X_i) g_{nm}(X_j))$ equals

$$EE(g_{nm}(X_i) g_{nm}(X_j)/\Theta) = E(E(g_{nm}(X_i)/\Theta)\, E(g_{nm}(X_j/\Theta)) = \\ = EE^2(g_{nm}(X_1)/\Theta)).$$

In this derivation we used, of course, the assumed properties of the sequence (1). From (15) it follows now that $\sum_i E g_{nm}^2(X_i)$ tends to 0 when $m, n \to \infty$. Then $E g_{nm}^2(X_1) \to 0$. This means that $f_n(X_1)$ is a *Cauchy* sequence in L_2. Since L_2 is complete, $f_n(X_1)$ converges, in norm, to a function $X \in L_2$. Going over to a subsequence, we may assume that $f_n(X_1)$ converges to X a.e. Thus there exists a null set $N \in \mathfrak{F}$ such that

$$f_n(X_1(\omega)) \to X(\omega), \quad (\omega \notin N).$$

Define g by $g = \limsup f_n$. Then $X(\omega) = \lim f_n(X_1(\omega)) = \limsup f_n(X_1(\omega)) = g(X_1(\omega)), (\omega \notin N)$. Define f to be equal to g except when g is infinite in which case the value $f(x)$ is set equal to zero. Then f is a *Baire* function and since X is finite:

$$X(\omega) = \lim f_n(X_1(\omega)) = f(X_1(\omega)), \quad (\omega \notin N). \tag{16}$$

Since $X \in L_2$, we have $f \in B_1$.
Admit, for a moment, that for each $i = 1, 2, \ldots, t$, we have

$$\lim_n f_n(X_i) = f(X_i) \quad \text{a.e.} \tag{17}$$

Then it follows that Y_n converges to $f(X_1)+\ldots+f(X_t)$ a.e. Since a subsequence of Y_n converges a.e. to Y, we have $Y=f(X_1)+\ldots+f(X_t)$ a.e. and finally $Y\in V$.

It remains to show that (17) is true. This follows intuitively from the fact that the variables $X_1, \ldots, X_t$ are equidistributed and that, by (16), this relation (17) is true for $i=1$. The argument can be formalized as follows. Putting $h_n=|f-f_n|$, (17) is equivalent to

$$P\Big(\bigcap_{k=1}^{\infty}\ \bigcup_{m=1}^{\infty}\ \bigcap_{n=m}^{\infty}\ X_i^{-1}\left(h_n^{-1}\left[0, 1/k\right]\right)\Big)=1,$$

or, denoting by E the following set on the real line

$$E=\bigcap_{k=1}^{\infty}\ \bigcup_{m=1}^{\infty}\ \bigcap_{n=m}^{\infty}\ h_n^{-1}\left[0, 1/k\right],$$

to

$$P\big(X_i^{-1}(E)\big)=1.$$

Since (17) is true for $i=1$ and since $X_1, \ldots, X_t$ are equidistributed, we have

$$1=P\big(X_1^{-1}(E)\big)=P\big(X_2^{-1}(E)\big)=\ldots=P\big(X_t^{-1}(E)\big),$$

as remained to be proved.

5. The General Credibility Premium

5.1. For each $X\in L_2$, we have

$$E(X/X_1, \ldots, X_t)=\mathrm{PRO}(X/W). \tag{18}$$

In particular, the general credibility premium equals

$$X_{t+1}^{G}=E(X_{t+1}/X_1, \ldots, X_t)=\mathrm{PRO}(X_{t+1}/W), \tag{19}$$

which states that this premium is the projection of X_{t+1} on W.

5.2. The *risk premium* is defined by

$$m_\Theta=E(X_{t+1}/\Theta). \tag{20}$$

From the conditional independence of $X_1, \ldots, X_{t+1}$, given Θ, follows that

$$m_\Theta=E(X_{t+1}/\Theta, X_1, \ldots, X_t),$$

and then, from general properties of conditional means:

$$E(m_\Theta/X_1,\dots,X_t) = E(E(X_{t+1}/\Theta,X_1,\dots,X_t)/X_1,\dots,X_t) = E(X_{t+1}/X_1,\dots,X_t).$$

Thus,

$$E(m_\Theta/X_1,\dots,X_t) = E(X_{t+1}/X_1,\dots,X_t), \tag{21}$$

or

$$\mathrm{PRO}(m_\Theta/W) = \mathrm{PRO}(X_{t+1}/W). \tag{22}$$

6. The Semilinear Credibility Premium (fixed f)

6.1. Let f be fixed in B_1. For the particular $f(x) = x$, we have considerations about the linear credibility premium.
For each $X \in L_2$, we define

$$\bar{E}(X/f(X_1),\dots,f(X_t)) = \mathrm{PRO}(X/V_f) \tag{23}$$

Then the semilinear credibility premium $\bar{X}^f_{t+1}$ equals

$$X^f_{t+1} = \bar{E}(X_{t+1}/f(X_1),\dots,f(X_t)) = \mathrm{PRO}(X_{t+1}/V_f). \tag{24}$$

6.2. Using (9) and (22) it is seen that, since $V_f \subseteq W$,

$$\bar{X}^f_{t+1} = \mathrm{PRO}(X_{t+1}/V_f) =$$

$$\mathrm{PRO}(\mathrm{PRO}(X_{t+1}/W)/V_f) = \mathrm{PRO}(\mathrm{PRO}(m_\Theta/W)/V_f) = \mathrm{PRO}(m_\Theta/V_f).$$

Thus

$$\bar{X}^f_{t+1} = \bar{E}(m_\Theta/f(X_1),\dots,f(X_t)) = \mathrm{PRO}(m_\Theta/V_f), \tag{25}$$

and $\bar{X}^f_{t+1}$ is as well the best approximation of m_Θ as of X_{t+1} of the form $a_0 + a_1 f(X_1) + \dots + a_t f(X_t)$.

6.3. By the arguments used in [2], *Bühlmann* or [5], *Jewell*, it is not difficult to prove that

$$\bar{X}^f_{t+1} = (E\,X_1 - Z_{t+1}\,Ef(X_1)) + \frac{Z_{t+1}}{t}(f(X_1) + \dots + f(X_t)), \tag{26}$$

where

$$Z_{t+1} = \frac{t\,\mathrm{COV}(f(X_1), X_2)}{\mathrm{VAR}\,f(X_1) + (t-1)\,\mathrm{COV}(f(X_1), f(X_2))}. \tag{27}$$

Here, and in similar cases herafter, the indices 1 and 2 could be replaced by other ones. We also have

$$\| m_\Theta - \bar{X}^f_{t+1} \|^2 = \mathrm{COV}(X_1, X_2) - Z_{t+1}\,\mathrm{COV}(f(X_1), X_2). \tag{28}$$

These results are explicitly derived in [3], *De Vylder*.
Formulae (27) and (28) can also be written

$$Z_{t+1} = \frac{t\,\mathrm{COV}(f_\Theta, m_\Theta)}{E\,\mathrm{VAR}(f(X_1)/\Theta) + t\,\mathrm{VAR}\,f_\Theta} \tag{29}$$

and

$$\| m_\Theta - \bar{X}^f_{t+1} \|^2 = \mathrm{VAR}\,m_\Theta - Z_{t+1}\,\mathrm{COV}\,(f_\Theta, m_\Theta), \tag{30}$$

where

$$f_\Theta = E(f(X_1)/\Theta). \tag{31}$$

6.4. The result (26) and the argument of 6.2. show that

$$\bar{X}^f_{t+1} = \mathrm{PRO}(X_{t+1}/V'_f) = \mathrm{PRO}(m_\Theta/V'_f). \tag{32}$$

7. The Optimal Semilinear Credibility Premium

7.1. For each $X \in L_2$, we define

$$E^*(X/X_1, \ldots, X_t) = \mathrm{PRO}(X/V). \tag{33}$$

Then the optimal semilinear credibility premium clearly is

$$X^*_{t+1} = E^*(X_{t+1}/X_1, \ldots, X_t) = \mathrm{PRO}(X_{t+1}/V). \tag{34}$$

This solves the problem of existence and unicity, as a point of L_2, of this premium. For an optimal $f^* \in B_1$ we have

$$X^*_{t+1} = f^*(X_1) + \ldots + f^*(X_t).$$

7.2. The optimal f^* is not necessarely unique. For example, if $0 \leq X_1 \leq 1$ with probability 1, the definition of f^* outside the interval [0, 1] does not matter.

7.3. Since $V \subseteq W$, the argument of 6.2. shows that

$$X_{t+1}^{*} = E^{*}(m_{\Theta}/X_1, \ldots, X_t) = \mathrm{PRO}(m_{\Theta}/V).$$

7.4. The optimal premium (34) is relative to all of B_1. For each subset $B \subseteq B_1$ such that $V_B = \{f(X_1)+\ldots+f(X_t)/f \in B\}$ is a closed linear subspace of L_2, we have a related optimal premium

$$X_{t+1}^{*B} = \mathrm{PRO}(X_{t+1}/V_B) = \mathrm{PRO}(m_{\Theta}/V_B).$$

We remark that for fixed $f \in B_1$, the semilinear premium $\bar{X}_{t+1}^{f}$ is the particular X_{t+1}^{*B} where

$$B = \{a+bf/a, b \in R\}.$$

Of course, for each B, X_{t+1}^{*} is always at least as close to X_{t+1} as X_{t+1}^{*B}.

7.5. Let us now turn back to the optimal premium X_{t+1}^{*} and look for the determination of f^{*}.
From (34), we have $X_{t+1} - X_{t+1}^{*} \perp V$. This gives successively, for every $f \in B_1$:

$$X_{t+1} - f^{*}(X_1) - \ldots - f^{*}(X_t) \perp f(X_1) + \ldots + f(X_t),$$

$$E\left[(X_{t+1} - f^{*}(X_1) - \ldots - f^{*}(X_t))\,(f(X_1) + \ldots + f(X_t))\right] = 0,$$

$$E\left[X_{t+1}(f(X_1) + \ldots + f(X_t))\right] = E\left[(f^{*}(X_1) + \ldots + f^{*}(X_t))\,(f(X_1) + \ldots + f(X_t))\right],$$

$$tE(X_{t+1}f(X_1)) = tE\left((f^{*}(X_1) + \ldots + f^{*}(X_t))f(X_1)\right),$$

$$E(X_2 f(X_1)) = E(f^{*}(X_1)f(X_1)) + (t-1)\,E(f^{*}(X_2)f(X_1)), \quad (35)$$

Using the general formula $E(Y) = EE(Y/X_1)$, this relation is equivalent to

$$E\left[f(X_1)\left(E(X_2/X_1) - f^{*}(X_1) - (t-1)E(f^{*}(X_2)/X_1)\right)\right] = 0. \quad (36)$$

By taking f such that

$$f(X_1) = E(X_2/X_1) - f^{*}(X_1) - (t-1)E(f^{*}(X_2)/X_1),$$

it is immediate that this $f(X_1)$ must be zero a. e., since (36) then shows that the mean value of its square is zero.
So the optimal f^{*} is solution of the equation

$$E(X_2/X_1) = f^{*}(X_1) + (t-1)\,E(f^{*}(X_2)/X_1), \text{ a.e.} \quad (37)$$

Conversely, if f^{*} is a solution of (37), then (36) is true for every $f \in B_1$, and the preceding derivation may be taken in the opposite direction to prove that f^* is optimal.

7.6. The fundamental relation (37) may be used in two directions:

1) A theoretical one.
If the variables X_1, X_2 have a joint density $p(x, y)$, then relation (37) gives

$$\int y\, p(x,y)\,dy = f^*(x)\int p(x,y)\,dy + (t-1)\int f^*(y)\,p(x,y)\,dy.$$

This is an integral equation of the type studied in the *Fredholm* theory. The research of exact solutions in the case of particular distributions may be of interest. It could show for example how the optimal and the linear premium differ from each other. Moreover, the knowledge of these solutions could lead to forms of semilinear credibility, easy to handle with, not differing much from optimal credibility.

2) A practical one.
Numerical methods for the solution of the preceding integral equation do exist. More simply, if X_1 can only assume a finite number of distinct values, say $x_0, x_1, \ldots, x_n$, then (37) becomes the linear system

$$\sum_{j=0}^{n} x_j p_{ij} = f_i^* \sum_{j=0}^{n} p_{ij} + (t-1)\sum_{j=0}^{n} f_j^* p_{ij}\,(i = 0, \ldots, n), \left[p_{ij} = P\,(X_1 = x_i, X_2 = x_j)\right]$$

where the f_i^* are the unknown quantities. With our modern computers, such a system can be solved for great values of n.
A first study in this direction is made in [4], *De Vylder, Fl.* and *Ballegeer, Y.*, where the optimal and the linear forecasts for the number of claims in an automobile insurance portfolio are compared. The difference, even in such a simple case ($n = 5$), is far from negligible.

8. Extensions to the Multidimensional Credibility Model

8.1. It will suffice to consider the two-dimensional case. We then work with a sequence $\Theta, N_1, X_1, N_2, X_2, \ldots,$ of random variables, where for $\Theta = \theta$ fixed, the couples $(N_1, X_1), (N_2, X_2), \ldots$ are conditionally independent and equidistributed.
The interpretation of X_i is the same as before and N_i is the total number of claims in the year i of a contract chosen at random in the portfolio.

All considered variables are defined on a basic probability space $(\Omega, \mathfrak{I}, P)$. It is assumed, once for all, that the functions considered hereafter, defined on Ω, are measurable and square integrable. The real functions f, ϕ, ... of real variables are supposed to be *Baire* functions.
The space L_2 is defined as before.
The positive integer t is fixed.

8.2. For fixed f, a function of two variables, we here define V_f as being the closed linear subspace of L_2

$$V_f = \{a_0 + a_1 f(N_1, X_1) + \ldots + a_t f(N_t, X_t) / a_0, \ldots, a_t \in R\}.$$

The semilinear credibility premium $\bar{X}^f_{t+1}$ is

$$\bar{X}^f_{t+1} = \bar{E}(X_{t+1} / f(N_1, X_1), \ldots, f(N_t, X_t)) = \mathrm{PRO}(X_{t+1}/V_f) = \mathrm{PRO}(m_\Theta / V_f), \quad (38)$$

where m_Θ is again $E(X_{t+1}/\Theta)$.

8.3. The formulae of 6.3. remain valid if $f(X_i)$ is everywhere replaced there by $f(N_i, X_i)$.

8.4. The space V of variables of the form

$$f(N_1, X_1) + \ldots + f(N_t, X_t),$$

where f varies, is linear and closed (same argument as in 4.3.).
The optimal semilinear credibility premium is then

$$X^*_{t+1} = E^*(X_{t+1}/(N_1, X_1), \ldots, (N_t, X_t)) = \mathrm{PRO}(X_{t+1}/V) = \mathrm{PRO}(m_\Theta/V). \quad (39)$$

We have

$$X^*_{t+1} = f^*(N_1, X_1) + \ldots + f^*(N_t, X_t)$$

for an optimal f^*. Similarly as in the one-dimensional case, a two-dimensional integral equation can be written down for f^*.

8.5. When certain supplementary hypotheses are made on the given portfolio, related semilinear credibility premiums may be defined.
Suppose, for example, that for each fixed θ, the total cost of the claims in the year i for the contract θ may be written

$$X_{\theta i} = C_{\theta i1} + C_{\theta i2} + \ldots + C_{\theta i N_{\theta i}} \; (= 0 \text{ if } N_{\theta i} = 0), \quad (40)$$

where $N_{\theta i}$ is the number of claims of that contract and where $C_{\theta i1}$, $C_{\theta i2}$, ..., are independent random variables with same distribution function $S_\theta(x)$ depending at most on θ. (This are the level 1 hypotheses in [5], *Jewell.* In [3], *De Vylder*, the portfolio is then called «composé».) The random variable $C_{\theta ij}$ is the cost of the *j*-th claim of the *i*-th year of the contract θ.
For a fixed real function ϕ of one variable, define

$$Y_{\phi,\theta i} = \phi(C_{\theta i1}) + \phi(C_{\theta i2}) + \ldots + \phi(C_{\theta i N_{\theta i}}). \tag{41}$$

For a contract chosen at random in the portfolio, this relation then defines a random variable $Y_{\phi,i}$. We may then consider the premium

$$\bar{X}_{\phi,t+1} = \bar{E}(X_{t+1}/Y_{\phi,1}, \ldots, Y_{\phi,t}),$$

which is the best linear approximation of X_{t+1} in terms of $Y_{\phi,1}, \ldots, Y_{\phi,t}$.

8.6. In the preceding case we may again look for an optimal ϕ in the sense of minimizing $\| X_{t+1} - \bar{X}_{\phi,t+1} \|$, or, which is the same, $\| m_\Theta - \bar{X}_{\phi,t+1} \|$.
We mention the following, intuitively quite evident, particular result. Suppose that the partial costs $C_{\theta ij}$ all have the same distribution function $S(x)$. (Level 3 assumption in [5], *Jewell*). Then an optimal ϕ is $\phi \equiv 1$.
We note that $\phi \equiv 1$ reduces to the case $Y_{1,i} = N_i$. In other words, if all partial costs have same distribution function, the optimal credibility premium of the form $\bar{X}_{\phi,t+1}$ is that one depending only on the previous claim numbers $N_1, \ldots, N_t$. As far as the hypothesis of the equidistribution of the partial costs is verified, this is of course an argument in favour of the credibility systems now used in several countries in automobile liability insurance.
To prove the result, we set

$$c_\phi = E\,\phi(C_{\theta ij}), \quad d_\phi^2 = \mathrm{VAR}\,\phi(C_{\theta ij}), \quad \lambda_\theta = E(N_{\theta i}).$$

Then, from (41):

$$E(Y_{\phi,\theta i}) = c_\phi\, \lambda_\theta,$$
$$\mathrm{VAR}(Y_{\phi,\theta i}) = c_\phi^2\, \mathrm{VAR}(N_{\theta i}) + \lambda_\theta d_\phi^2.$$

Going over to the random variable Θ, these relations give

$$E(Y_{\phi,i}/\Theta) = c_\phi\, \lambda_\Theta,$$
$$\mathrm{VAR}(Y_{\phi,i}/\Theta) = c_\phi^2\, \mathrm{VAR}(N_i/\Theta) + d_\phi^2\, \lambda_\Theta. \tag{42}$$

With the hypotheses of the preceding section 3.5., (29) and (30) are to be written

$$Z_{t+1} = \frac{t\,\mathrm{COV}(m_{\phi,\Theta}, m_\Theta)}{E.\mathrm{VAR}(Y_{\phi,1}/\Theta)+t\,\mathrm{VAR}(m_{\phi,\Theta})},$$

$$\| m_\Theta - \bar{X}_{\phi,t+1} \| = \mathrm{VAR}(m_\Theta) - Z_{t+1}\,\mathrm{COV}(m_{\phi,\Theta}, m_\Theta),$$

where

$$m_{\phi,\Theta} = E(Y_{\phi,1}/\Theta) = c_\phi\,\lambda_\Theta. \tag{43}$$

For $\phi \equiv 1$:

$$m_{1,\Theta} = E(N_1/\Theta) = \lambda_\Theta.$$

So we have to prove that

$$\frac{\mathrm{COV}^2(\lambda_\Theta, m_\Theta)}{E.\mathrm{VAR}(N_1/\Theta)+t\,\mathrm{VAR}(\lambda_\Theta)} \geq \frac{\mathrm{COV}^2(m_{\phi,\Theta}, m_\Theta)}{E.\mathrm{VAR}(Y_{\phi,1}/\Theta)+t\,\mathrm{VAR}(m_{\phi,\Theta})}.$$

Using (42), (43) and simplifying by c_ϕ^2, the last expression reduces to

$$\frac{\mathrm{COV}^2(\lambda_\Theta, m_\Theta)}{E.\mathrm{VAR}(N_1/\Theta)+E(\lambda_\Theta)\dfrac{d_\phi^2}{c_\phi^2}+t\,\mathrm{VAR}(\lambda_\Theta)}.$$

Owing to the presence of the term $E(\lambda_\Theta)\ d_\phi^2/c_\phi^2$ in the denominator, the assertion is proved.

Bibliography

[1] *Bühlmann, H.:* Experience rating and credibility. The ASTIN Bulletin, July 1967.

[2] *Bühlmann, H.:* Mathematical methods in risk theory. Springer-Verlag, 1970.

[3] *De Vylder, Fl.:* Introduction aux théories actuarielles de crédibilité. Office des Assureurs de Belgique, 1975.

[4] *De Vylder, Fl., Ballegeer, Y.:* A numerical illustration of optimal semilinear credibility. Presented at the ASTIN colloquium 1975.

[5] *Jewell, W.S.:* Multi-dimensional credibility. University of California, Berkeley, August 1973.

[6] *Thyrion, P.:* Quelques observations statistiques sur la variable "nombre de sinistres" en assurance automobile. The ASTIN Bulletin, May 1972.

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Zusammenfassung

Lineare Credibility-Formeln sind beste lineare Approximationen von bedingten Erwartungswerten $E[X_{t+1} | X_1, X_2, \ldots, X_t]$, *semilineare* Credibility-Formeln werden genau gleich definiert, jedoch für allgemeinere Erwartungswerte der Form $E[X_{t+1}\ f(X_1), f(X_2), \ldots, f(X_t)]$. Zunächst werden den altbekannten Credibility-Faktoren formal entsprechende Faktoren hergeleitet und anschliessend optimale Funktionen $f(x)$ gesucht, was auf Fredholmsche Integralgleichungen führt.

Résumé

Les formules de credibility *linéaires* sont les approximations linéaires pour des espérances mathématiques $E[X_{t+1} | X_1, X_2, \ldots, X_t]$. Les formules sémilinéaires résultent de la même règle sauf que l'on considère des espérances conditionnelles plus générales du type $E[X_{t+1}\ f(X_1), f(X_2), \ldots, f(X_t)]$. D'abord, les facteurs de credibility sémilinéaire aux facteurs «classiques» sont déduits et l'article s'occupe de la détermination d'une fonction $f(x)$ optimale, ce qui amène à des équations intégrales du type Fredholm.

Riassunto

Le formole *lineari* di credibility sono definite come approssimazioni lineari di speranze matematiche $E[X_{t+1} | X_1, X_2, \ldots, X_t]$. Le formole *semilineari* seguono la stessa definizione colla differenza che le speranze matematiche sono generalizzate a $E[X_{t+1}\ f(X_1, f(X_2), \ldots, f(X_t)]$. Nell'articolo si derivano dapprima fattori di credibility che correspondono ai fattori classici. Nella seconda parte dell'articolo la determinazione delle funzioni $f(x)$ ottimali conduce alla teoria delle equazioni integrali di Fredholm.

Summary

Linear credibility formulae are by definition equal to the least square linear approximation of a conditional expectation $E[X_{t+1} | X_1, X_2, \ldots, X_t]$. For *semilinear* formulae the same definition holds except that more general expectations namely $E[X_{t+1}\ f(X_1), f(X_2), \ldots, f(X_t)]$ are considered. First, credibility factors corresponding to the classical ones are deduced. Then, in a second part, the search for an optimal function $f(x)$ leads to integral equations of the Fredholm type.