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Trends in the Numerical Analysis of Nonlinear Structural Problems

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A review is given of the historic development of finite element analysis methods for nonlinear structural problems. The shortcomings of these more classical techniques for current strongly nonlinear analysis requirements are discussed. In particular, attention is paid to analysis of problems of large strain and deformation, to analysis of nonlinear (post-)buckling problems and to the nonlinear dynamic analysis of large structures.

The difficulties that occur in the above problem areas are, amongst others, excessive computational cost. Procedures to overcome these difficulties are currently under development in many institutions. The current state of these procedures is described, and some indication is given to which extent these new techniques will eventually enlarge the capabilities for numerical analysis of nonlinear structures.

Es wird eine Übersicht der geschichtlichen Entwicklung der Finite-Elemente-Methode für nichtlineare Strukturprobleme gegeben. Unzulänglichkeiten eher klassischer Verfahren für geläufige, stark nichtlineare Berechnungen werden gezeigt. Besondere Aufmerksamkeit werden der Berechnung von Problemen mit grosser Dehnung und Deformation, der Berechnung von nichtlinearen (Nach-)Knickproblemen und der nichtlinearen dynamischen Analyse grosser Strukturen geschenkt.

Die Schwierigkeiten, die in obigen Problemkreisen auftreten, sind, nebst anderen, übermässige Rechenkosten. Verfahren, die diese Schwierigkeiten zu umgehen versuchen, sind in vielen Instituten laufend in Entwicklung. Es wird der momentane Stand dieser Verfahren erläutert. Weiter werden einige Angaben über das Ausmass möglicher Erweiterungen numerischer Berechnungen von nichtlinearen Strukturen, mit diesen neuen Verfahren, dargelegt.

Traditional Solution Methods for Nonlinear Problems

After the introduction of the finite element method for linear problems, it was soon discovered that the method could also be used successfully for, up to that point, *unsolvable nonlinear problems.* This was not only interesting as an academic exercise: there definitely existed a need for results of nonlinear analysis in industry. In the first decade of the development of nonlinear finite element technology, the direction of development was strongly influenced by the needs of nuclear and aerospace industry [1, 2].

In the *nuclear industry,* nonlinear analysis of *nuclear components* has to be carried out where nonlinearities are primarily due to nonlinear high temperature material behavior. Occasionally *geometric nonlinearities* have to be included, but they are often of a secondary nature. At the same time the *aerospace industry,* for which initial requirements were primarily concerned with linear analysis, was also requesting nonlinear analysis capabilities. In the structural area the main nonlinear phenomena to be considered are *geometric in nature (buckling),* whereas in the *jet engine technology* the problems are very similar to those in the nuclear industry. Over the years, several commercially available finite element codes have been developed to satisfy these demands.

The nonlinear finite element analysis capabilities were originally developed from linear finite element programs. Hence, the approach taken to solve these nonlinear problems usually was derived from the methods used for linear problems. One such approach is the initial *stress/strain approach for plasticity problems.* Here all nonlinear effects are included on the right hand side of the system of linear finite element equations, which requires an iterative procedure to solve the nonlinear problems (Figure 1a). Another popular approach to the solution of nonlinear problems is to consider the nonlinear problem as a sequence of linear problems. Here the approach is to solve the instantaneous problem, advance the solution and solve the next instantaneous problem (Figure 1b).

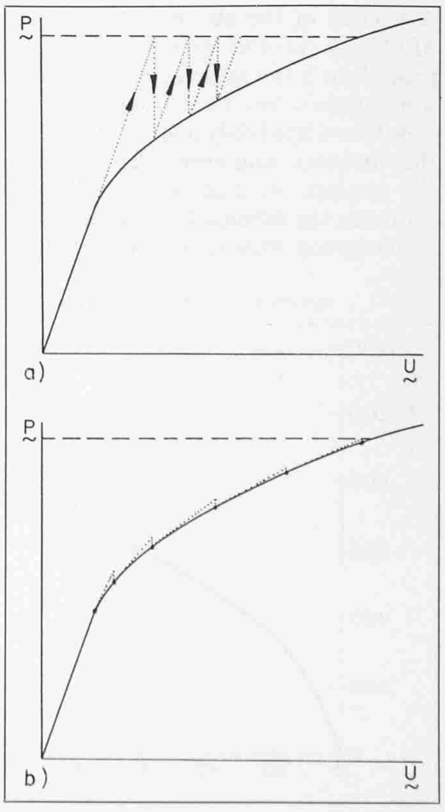
Note that in this approach a new linear system of equations needs to be formulated and solved in each step.

The "modified linear" approaches discussed above usually perform satisfactorily for problems with *"mild" nonlinearities.* Problems of this type are for instance elastic-plastic problems with a small plastic zone, and geometric nonlinear problems prior to snap-through or buckling. However, if the nonlinearity increases these simple methods often produce poor results or fail to produce results at all. More sophisticated approaches are needed under those circumstances, and will be discussed in the remainder of this paper.

Current Approaches to Nonlinear Problems

In order to solve severely nonlinear problems, it is no longer useful to consider these problems as modified linear problems. Instead, it is better to consult the literature to see which methods mathematicians have devised to solve large systems of nonlinear equations. A very important aspect of considering the nonlinear finite element problem along this line is that the system *stiffness matrix* is no longer all important. Instead the central point in nonlinear analysis methods is the *evaluation of the nodal equilibrium,* and the *reduction of errors in the equilibrium.* In fact, if possible one would like to avoid formation and solution of a stiffness matrix altogether considering the cost associated with it. On the other hand, one does not want to create a method which needs an excessive amount of iterations, since this will increase the cost as well. Above all one desires a method which is reliable, in the sense that it produces a (correct) so-

Figure 1. Classical analysis schemes for nonlinear problems



* Vgl. Schweizer Ingenieur und Architekt, Heft 51/52: 1117–1121, 1982; Heft 1/2: 2–7; Heft 4: 42–46, 47–50; Heft 9: 275–278, 279–281; Heft 15: 409–412, 1983.

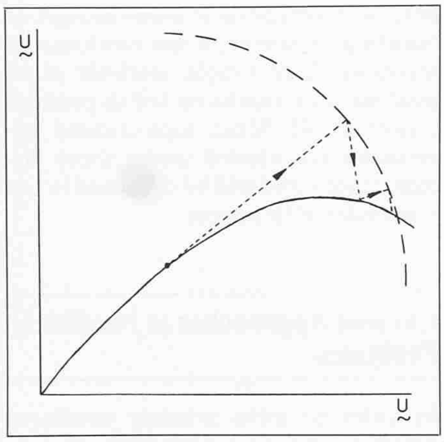


Figure 2. Constrained iteration in a two-dimensional space

lution in all circumstances, even if the cost is not quite optimal.

Most solutions methods which are currently investigated are variants of the *Newton method*. The classical (Full) Newton Method, which is used successfully in many finite element programs, seeks to eliminate the equilibrium errors by linearizing the nonlinear governing equations around the last obtained approximate solution.

The disadvantage of the Full Newton Method is that it requires formation and solution of the stiffness matrix in every increment. In return for that, it yields quadratic convergence, that is once one approaches the true solution, the relative error in the next approximation is proportional to the square of the error in the current approximation. Hence, usually only very few iterations are needed with this method.

A variant of the above method is the *Modified Newton Method;* the basic procedure is the same as for Full Newton iteration, but the stiffness matrix is now formulated only once at the start of the iteration sequence. Per iteration, the savings in cost is considerable: however, the method loses its quadratic convergence properties, and particularly for strongly nonlinear problems converges very poorly or not at all.

Most recent research on this topic attempts to develop methods which combine the advantages of both Full and Modified Newton Methods. These methods have a variety of associated names such as *Quasi-Newton, Conjugate Newton* and *Secant-Newton* and they seek to modify the inverse of the stiffness matrix and/or the right-hand side of the system directly in order to speed up the convergence of the solution process. In addition, so-called *"line searches"* are sometimes carried out in order to determine the optimal size of the solution correction. With such procedures, remarkable improvements have been obtained for certain classes of problems. However, instances in which these modifications had no beneficial effects are also known. The interested reader is referred to [3] for a more detailed discussion.

Another line along which the solution of nonlinear problems is explored is the so-called *reduced basis technique*. By only considering certain solution modes, the large system of equations is first transformed to a (very) small system of equations, which is then readily solved. Such methods are quite successful if, from an existing solution, one can derive the main participating solution modes, and these modes stay about the same during a fair portion of the loading history [4]. However, if the solution pattern changes dramatically a new reduced basis must be calculated, which requires solution of the large system of equations, and eliminates the advantages of the procedure.

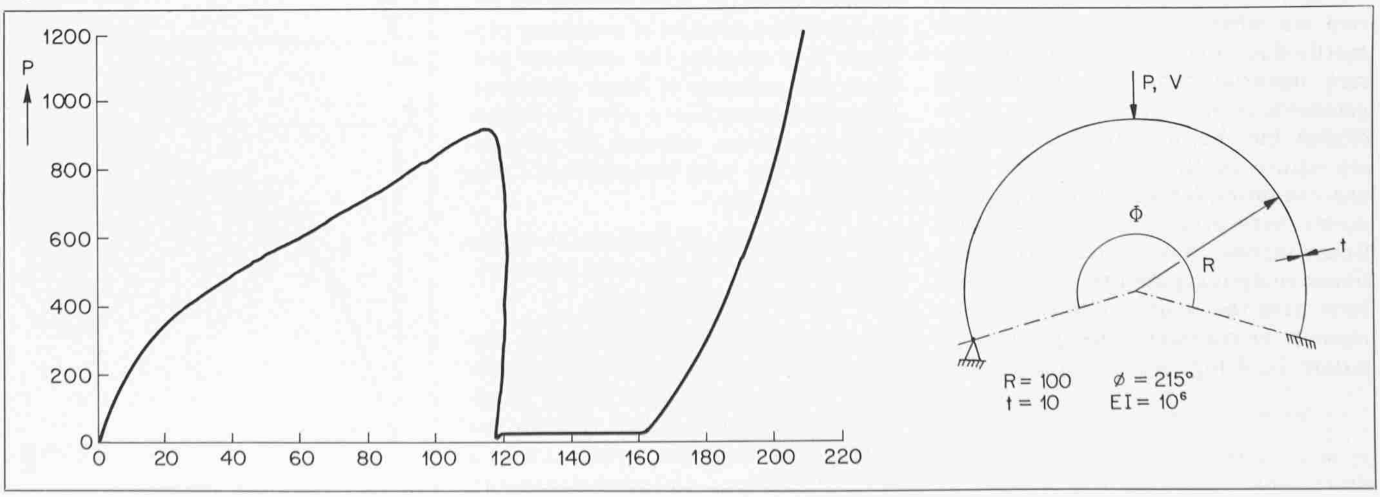
As a final note, it may be observed that a nonlinear system of equations does not always immediately exist in algebraic form. In particular in plasticity, viscoelasticity, creep, dynamics, etc., a finite element idealization would yield a *system of (nonlinear) differential equations*. This so-called *semi-discreet system* must then first be integrated in a suitable manner. In the past, simple explicit integration procedures were often used for this purpose. Such procedures usually have definite stability limits and hence require large numbers of load/time increments. For dynamics, more sophisticated implicit procedures have been used successfully for some time. For plasticity, creep and viscoelasticity, the realization that implicit procedures provide considerable advantages has only transpired recently.

Quasi-Static Post-Buckling Analysis

Problems with geometric nonlinearities have been analyzed successfully with the finite element method for many years. However, classical analysis procedures only yield satisfactory results prior to the occurrence of instable phenomena. If snap-through or buckling phenomena occur, classical solution procedures fail to give results. In order to obtain solutions for such problems, it is necessary to control the *magnitude of the incremental solution* and hence the *magnitude of the load increment automatically*. Such techniques were originally proposed by *Riks* [5] and *Wempner* [6], but had the practical problem that the usual bandedness of the system of finite element equations was destroyed. Later, the method was modified to overcome this difficulty by doing the adjustment of the load step separately after solution of the system of equations.

In this method one requires that a norm of the displacement increment is equal to a certain value. Usually one prescribes the Euclidian norm which leads to an iteration procedure "on a sphere". For a two-dimensional problem, such an iteration procedure is shown in Fi-

Figure 3. Asymmetric arch snap-through and buckling



gure 2. Note that this "constrained" iteration procedure, which features simultaneous iteration on loads and displacements, can be used with all variants of the Newton Method. An example of the power of this method calculated with a development version of the MARC program [7] is shown in Figure 3.

An arch shown in Figure 3 is hinged at one end and simply supported at the other end. A point load is applied at the center of the arch, and the solution is calculated for the load range from 0 to 1200. The solution for the initial phase of the loading process is presented in [8], and agrees well with the solution obtained here. Note that the problem has some peculiar features. In particular it has an initial snap-through phenomena, followed by an asymmetric buckling phase where the displacement stays approximately constant with a decreasing load. This makes it almost impossible to obtain a solution with traditional displacement control.

Finite Strain Plasticity Problems

In the use of the finite element method for analysis of large strain plasticity problems (such as metal forming problems), important differences arise as compared to classical elastic-plastic problems. One is the fact that the constitutive equations must be written in an appropriate form. The form usually chosen is one where the *Jaumann rate of Cauchy stress* is a linear function of the deformation rate, as described in [9]. The inclusion of the extra terms associated with this in a finite element scheme does not present any computational difficulties. The other difference is more of a practical nature, namely, that the plastic strains are usually orders of magnitude larger than in classical applications.

In the classical solution approach, one would formulate the rate equations at the beginning of an increment, solve these (linear) rate equations and advance the solution for a certain step size. This clearly is an explicit method, and unfortunately such an explicit method has a definite stability limit. In fact, it can be proven that local stability (on the integration point level) is only obtained if the plastic strain increment is less than twice the elastic strain [10]. For typical applications, the elastic strains are of the order of 0,1% and the desired plastic strains are 100% or more. Hence, with a classical approach the number of increments will be in the order of 1000, with associated prohibitive computing cost. Clearly more advanced forward integration procedures are desired.

A natural approach seems to be to start from a formulation based on a finite increment in the solution. Then an assumption is made about the solution path within the increment, and the rate equations are integrated to a set of (nonlinear) incremental equations. After that, an appropriate solution procedure as discussed in section 2 can be applied to solve the set of equations. This approach (with the assumption of a straight strain path within an increment) has been worked out in [11], and applied successfully to a number of problems. Increments of plastic strain in the order of 10 to 20% turned out to be feasible, and although some iterations are necessary to solve the nonlinear equations, cost savings of a factor of 10 or more are readily obtained.

An example is shown in Figure 4 (calculated with the MARC program [7]). A cylindrical disk is bonded to a rigid punch, and is decreased in height by 44%. The dimension and properties of the specimen are shown in Figure 4a. The analysis was carried out in 44 increments, with strain increments larger than 10% towards the end of the analysis. The load deflection curve and the deformed mesh at the end of the analysis are shown in figures 4b and 4c. Note that the jump in the load-deflection curve occurs when the first node on the outside boundary comes in contact with the punch and hence is due to the somewhat crude modeling and not a consequence of the solution procedure. That the results were indeed independent of the increment size was established by repeating the analysis with a five times smaller load step. The differences in solution were less than 1%.

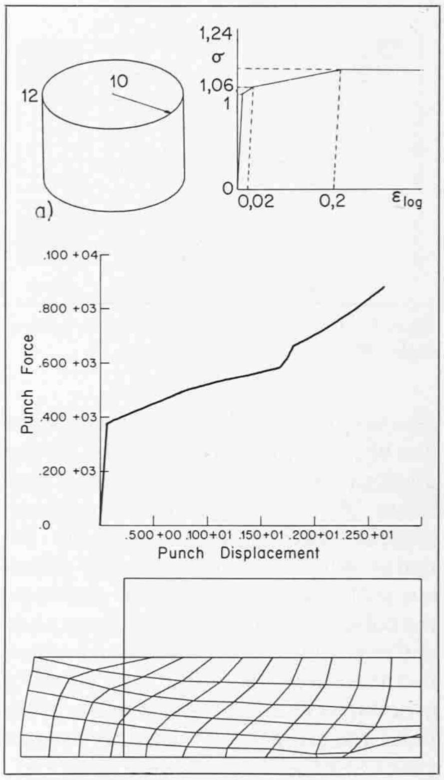


Figure 4. Upsetting of a cylindrical disk

Nonlinear Dynamics Problems

If a finite element discretization of a structural dynamics problem is made, one arrives at a set of coupled nonlinear differential equations, the so-called *semi-discreet system.* To ultimately solve this problem, one has to integrate these equations with respect to time with use of some discreet integration operator. Two types of classical approaches exist to carry out this task: explicit methods and implicit methods.

In the *explicit method,* one calculates the accelerations based on the nodal force imbalance at a given point in time, and assumes that these accelerations remain constant during a finite increment. This method only requires one single inversion of the mass matrix during the analysis which is a trivial effort, particularly if the mass matrix is diagonalized. However, the main drawback of the explicit method is its limited stability. The maximum stable time step is related to the period of the highest eigenfrequency in the structure. Particularly, for large structures with refined element meshes this often leads to prohibitively small time steps.

In contrast, *implicit methods* require an inversion of an operator matrix which is a linear combination of mass and (tangent) stiffness matrix. For linear problems, such methods can be selected to have very favorable characteristics. One of the most popular methods is the *Newmark-Beta Method,* which in linear problems is unconditionally stable and has no artificial damping. For nonlinear problems, the unconditional stability cannot be guaranteed any more, but for most analyses this does not actually present a problem. The main disadvantage of the method, however, is the need for inversion of an operator matrix which includes the stiffness matrix. Due to nonlinearities, the stiffness matrix changes in time which makes (very) frequent formation and inversion of the operator matrix necessary. For large problems, this often completely offsets the advantage over explicit methods.

Recent research in nonlinear dynamics has concentrated on developing methods which combine the advantages of both methods. One approach which has been employed successfully is the im-

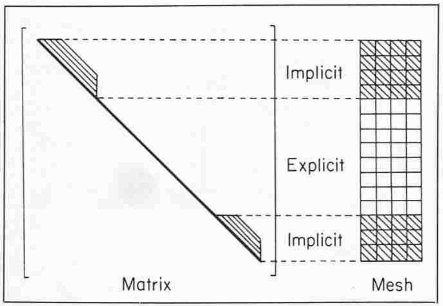
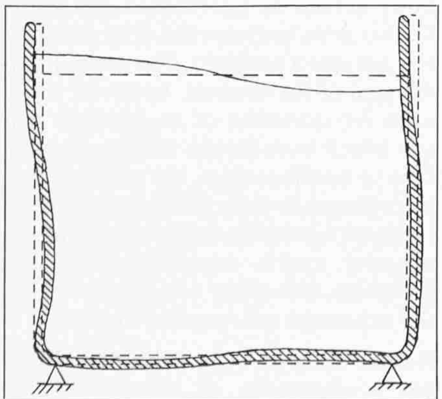


Figure 5. Operator matrix profile in an implicit-explicit method

plicit-explicit method. This approach has been particularly successful in the *analysis of coupled fluid-solid problems.* Here the finite element mesh represents two different physical domains, which have considerably different stiffnesses. In fluid-solid problems the solid is usually orders of magnitude stiffer than the fluid, and in an explicit method lowers the stable time step to a prohibitively small value. On the other hand, the presence of the fluid elements creates a very large system of equations.

Such problems have now been successfully solved by integrating the fluid with an explicit and the structure with an implicit method. Originally the explicit-implicit splitting was done on a nodal level [12], which caused complications on the interface. Later it was demonstrated that the method could be made much more effective by doing the explicit-implicit splitting on the element level [13]. Different procedures are then used to form the operator matrices for the individual elements. With a diagonalized mass matrix the profile of the total operator matrix then becomes very suitable for treatment with a modern "skyline" type equation solver (Figure 5). With such techniques, problems like sloshing of fluid in a tank which undergoes fairly large distortions can be solved effectively (Figure 6). Though implicit-explicit techniques offer considerable advantages in coupled problems, they are of limited effectivity in structures in which large stiffness differences do not exist. Research is currently underway to develop new approaches to the nonlinear dynamic problem.

Figure 6. Sloshing of a fluid in a tank



In general these techniques try to find an operator matrix which does not have the full bandwidth of the stiffness matrix, but in a narrow diagonal band contain enough stiffness information to increase stability considerably. Such techniques may be called *semi-implicit* [14]. A specific variant of this family of methods is the *operator-split technique,* in which the (implicit) operator matrix is approximated by product of very easily invertible matrix factors [15]. As yet, insufficient experience exists with these techniques to ascertain whether they really form a large improvement in dynamic analysis of nonlinear structures.

Conclusions

In recent research on finite element analysis procedures, considerable advances have been made in the treatment of strongly nonlinear problems. Due to these advances, it has now become possible to solve heretofore unsolvable instable post-buckling problems, and to treat large strain plasticity problems with little more cost than classical elastic-plastic problems. Advances in many other areas of application have been made leading again to extended and more efficient analysis capabilities. For a more complete review of these recent developments, the interested reader is referred to [16].

It is the author's opinion that in the near future considerable further progress in development of nonlinear finite element analysis procedures will be made. Such improved procedures will:

- solve currently unsolvable problems;
- treat very large nonlinear problems more efficiently;
- have controls for automatic load or time incrementation.

In particular, the last aspect is of paramount importance in order to achieve more widespread use of nonlinear finite element analysis procedures for practical engineering problems.

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